SMOOTH FINITE ELEMENTS AND STABLE SPLITTING*

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Abstract. We present error bounds for the approximation from the spaces of multivariate piecewise polynomials admitting stable local bases. In particular, the bounds apply to the spaces of smooth finite elements in n variables. In addition, in the case of a space of bivariate quintic C^1 piecewise polynomials we discuss its stable splitting into a subspace satisfying homogenious boundary conditions and its complement. These results are used by K. Böhmer [3] in his finite element method for general fully nonlinear elliptic differential equations of second order.

1. Introduction. Multivarite smooth piecewise polynomials have long been studied in the finite element method, see e.g. the famous Ciarlet's book [5]. Clearly, in many situations low order non-conforming or even discontinuous elements are preferable. However, it seems difficult to apply the approaches relying on these elements in the case when no weak variational formulation of the differential equation is available, for example for the fully nonlinear elliptic equations. Therefore, the first general finite element method for these equations developed by K. Böhmer [3] is based on smooth finite elements.

The theory of such elements, especially in two variables, has been significantly developed in recent years for the needs of approximation, see [13]. In particular, nested spaces of smooth finite elements and C^1 hierarchical bases are available [7, 11].

However, certain important questions have not been addressed, in particular the error bounds for multivariate smooth piecewise polynomials on general triangulations in $\mathbb{R}^n$. The first goal of this paper is to fill in this gap, see Theorem 3.1. K. Böhmer [3] relies on this result in his proof of the error bounds for his finite element method for fully nonlinear differential equations. Our second major goal is to provide a new construction of a modified Argyris finite element that allows a stable splitting of the finite element space into a subspace satisfying homogenious boundary conditions and its complement, see Section 5. This construction is motivated by the needs of the finite element method presented in [3].

Let $\mathcal{T}$ be a *triangulation* of a bounded polyhedral domain $\Omega \subset \mathbb{R}^n$, *i.e.*, $\mathcal{T}$ is a finite set of non-degenerate n-simplices such that

- $\Omega = \bigcup_{T\in\mathcal{T}} T$;
- the interiors of the simplices in $\mathcal{T}$ are pairwise disjoint; and
- each facet of a simplex $T \in \mathcal{T}$ either lies on the boundary of Ω or is a common face of exactly two simplices in $\mathcal{T}$.

Let v be a vertex of $\mathcal{T}$. The *star* of v, denoted by $\mathrm{star}(v)$, is the union of all n-simplices $T \in \mathcal{T}$ attached to v. We set $\mathrm{star}^1(v) := \mathrm{star}(v)$, and define $\mathrm{star}^\gamma(v)$, $\gamma \geq 2$, recursively as the union of the stars of all vertices of $\mathcal{T}$ contained in $\mathrm{star}^{\gamma-1}(v)$.

Let us denote by $|\mathcal{T}|$ the maximum diameter of simplices $T \in \mathcal{T}$, and by $\omega_\mathcal{T}$ the *shape regularity constant* of the triangulation $\mathcal{T}$,

$$\omega_\mathcal{T} := \max_{T\in\mathcal{T}} \frac{h_T}{\rho_T},$$

where h_T and ρ_T are the diameter of T and the diameter of its inscribed sphere, respectively.

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For any $d \geq 0$, let $S_d(\mathcal{T})$ denote the space of all piecewise polynomial functions w.r.t. $\mathcal{T}$, *i.e.*,

$$s \in S_d(\mathcal{T}) \quad \Longleftrightarrow \quad s|_T \in \Pi_d^n \ \text{ for all } n\text{-simplices } T \in \mathcal{T},$$

where Π_d^n is the linear space of all n-variate polynomials of total degree at most d. It is well-known that $\dim \Pi_d^n = \binom{n+d}{n}$.

Our goal is to provide upper bounds for the L_p approximation error from any subspaces $S \subset S_d(\mathcal{T})$, such that the constants in the estimates depend only on $n, p, d, \omega_{\mathcal{T}}$, and some parameters of a basis for S characterizing its *stability* and *locality*, see Theorem 3.1 below. In the case of finite element subspaces these bounds can be found e.g. in [4], and in the bivarite setting in [12]. Anisotropic triangulations have been considered in [8].

Let $C^r(\Omega)$ denote the linear space of all r times continuously differentiable functions on Ω. We are particularly interested in the spaces of smooth piecewise polynomials (*splines*)

$$S_d^r(\mathcal{T}) := S_d(\mathcal{T}) \cap C^r(\Omega), \qquad 1 \leq r < d,$$

and their subspaces. There are various constructions of stable local bases for $S_d^r(\mathcal{T})$ and/or certain subspaces thereof, see discussion in Section 4 below. The error bounds of Theorem 3.1 apply to all spaces where such bases are available, in particular to the full spline spaces $S_d^r(\mathcal{T})$ on arbitrary triangulations as soon as $d \geq r2^n + 1$.

We will use the following notation for the function spaces and norms. For any domain $G \subset \mathbb{R}^n$ we consider the L_p-spaces with the norm

$$\|f\|_{L_p(G)} = \begin{cases} (\int_G |f(x)|^p \, dx)^{1/p}, & 1 \leq p < \infty, \\ \operatorname{ess\,sup}_{x \in G} |f(x)|, & p = \infty, \end{cases}$$

as well as the *Sobolev spaces* $W_p^\mu(G)$, $\mu = 1, 2, \ldots$, with the semi-norm

$$|f|_{W_p^\mu(G)} = \begin{cases} (\sum_{|\alpha|=\mu} \|D^\alpha f\|_{L_p(G)}^p)^{1/p}, & 1 \leq p < \infty, \\ \max_{|\alpha|=\mu} \|D^\alpha f\|_{L_\infty(G)}, & p = \infty, \end{cases}$$

where D^α denotes the usual pointwise partial derivative

$$D^\alpha := \Big(\frac{\partial}{\partial x_1}\Big)^{\alpha_1} \cdots \Big(\frac{\partial}{\partial x_n}\Big)^{\alpha_n}, \qquad |\alpha| := \alpha_1 + \cdots + \alpha_n,$$

whenever appropriate, and a weak derivative otherwise. Note that in case $\mu = 0$ we have

$$|f|_{W_p^0(G)} = \|f\|_{L_p(G)}, \qquad 1 \leq p \leq \infty,$$

and that $H^\mu(G)$ is alternative notation for the space $W_2^\mu(G)$.

Given a triangulation $\mathcal{T}$ of Ω, we also consider the *mesh-dependent* L_p norm defined by

$$\|f\|_{L_p(\omega)}^{\mathcal{T}} = \begin{cases} \Big(\sum_{T \in \mathcal{T}} \|f\|_{L_p(T)}^p\Big)^{1/p}, & 1 \leq p < \infty, \\ \max_{T \in \mathcal{T}} \|f\|_{L_\infty(T)}, & p = \infty. \end{cases}$$

The error bounds below are formulated for any triangulation $\mathcal{T}$, with explicit mentioning of the parameters on which the constants in an estimate depend. Clearly, if one generates a family of triangulations $\{\mathcal{T}^h\}$ and spaces $\{S^h\}$ parametrised by some parameter h, for example $h = |\mathcal{T}|$, and assumes that the other parameters, such as $n, d, p, \omega_{\mathcal{T}}$ remain fixed or bounded, then one obtains respective asymptotic estimates and rates of convergence.

In particular, the assumption that $\omega_{\mathcal{T}^h}$ is bounded is equivalent to saying that the family $\{\mathcal{T}^h\}$ is *non-degenerate*, see [3].

The paper is organized as follows. After briefly discussing inverse estimates in Section 2, we prove the main error bounds in Section 3. Section 4 is devoted to particular examples of spaces of smooth piecewise polynomials to which the bounds apply, where we concentrate on Argyris finite element. Section 5 provides a splitting construction needed in [3].

2. Inverse Estimates. We start by establishing the inverse estimates that will be needed below in the proof of the error bound and are, in the same time, of independent interest in the finite element method, e.g. in [3].

The *multivariate Markov inequality* [6] for a simplex $T \subset \mathbb{R}^n$,

$$\|D^\alpha s\|_{L_\infty(T)} \le c\frac{nd^2}{\rho_T}\|s\|_{L_\infty(T)}, \qquad s \in \Pi_d^n, \qquad |\alpha| = 1,$$

where c is an absolute constant, implies the following *inverse estimates.*

THEOREM 2.1. *For any $1 \le p \le \infty$, $0 \le k < \mu \le d$, we have*

$$|s|_{W_p^\mu(T)} \le \frac{A}{h_T^{\mu-k}}|s|_{W_p^k(T)}, \qquad s \in S_d(\mathcal{T}), \quad T \in \mathcal{T}, \tag{2.1}$$

where the constant A depends only on $n, d, p, \omega_{\mathcal{T}}$.

Proof. Let $|\alpha| = \mu$ and $g = s|_T \in \Pi_d^n$. Since $D^\alpha g = D^\beta D^\gamma p$ for some β, γ with $|\beta| = \mu - k, |\gamma| = k$, and since $D^\gamma g \in \Pi_{d-k}^n$, a repeated application of Markov inequality infers

$$\|D^\alpha g\|_{L_\infty(T)} \le \frac{a_1}{\rho_T^{\mu-k}}\|D^\gamma g\|_{L_\infty(T)} \le \frac{a_2}{h_T^{\mu-k}}\|D^\gamma g\|_{L_\infty(T)},$$

where a_1 depends only on n, d, and $a_2 = a_1\omega_{\mathcal{T}}^{\mu-k}$. This already proves (2.1) in the case $p = \infty$. For $1 \le p < \infty$, a simple scaling argument shows that

$$a_3^{-1}\text{vol}^{1/p}(T)\|q\|_{L_\infty(T)} \le \|q\|_{L_p(T)} \le a_4\text{vol}^{1/p}(T)\|q\|_{L_\infty(T)}, \qquad q \in \Pi_d^n, \tag{2.2}$$

where $\text{vol}\,(T)$ is the n-dimensional volume of the simplex T, and the constants a_3, a_4 depend only on n, d, p. Therefore, we get

$$\|D^\alpha g\|_{L_p(T)} \le \frac{a_2 a_3 a_4}{h_T^{\mu-k}}\|D^\gamma g\|_{L_p(T)},$$

and (2.1) follows. ■

3. Approximation bounds. Let S be a linear subspace of $S_d(\mathcal{T})$, and let $\{s_1, \dots, s_m\}$ be a basis for S. Suppose that $\{\lambda_1, \dots, \lambda_m\} \subset S^*$ is its dual basis, i.e.

$$\lambda_i s_j = \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The following theorem provides error bounds for certain *quasi-interpolation* operator $Q : L_1(\Omega) \to S$. Note that in the applications to the finite element method usually only the *existence* of an operator Q with desired approximation properties is important. Hence it is acceptable to use such tools as Hahn-Banach Theorem in the definition of Q.

THEOREM 3.1. *Suppose that for each $k = 1, \ldots, m$, there is a set $E_k \subset \Omega$ such that*

$$E_k \subset \operatorname{star}^\gamma(v_k) \quad \textit{for an appropriate vertex } v_k, \tag{3.1}$$

$$\operatorname{supp} s_k \subset E_k, \tag{3.2}$$

$$\|s_k\|_{L_\infty(\Omega)} \le C_1, \tag{3.3}$$

and

$$|\lambda_k s| \le C_2 \|s\|_{L_\infty(E_k)}, \quad \textit{for all } s \in S, \tag{3.4}$$

for some C_1, C_2 and γ. Moreover, assume that

$$\Pi^n_{\ell-1} \subset S \quad \textit{for some } 1 \le \ell \le d+1. \tag{3.5}$$

Then there exists a linear operator $Q : L_1(\Omega) \to S$, such that for any $T \in \mathcal{T}$, $1 \le p \le \infty$, $0 \le |\alpha| \le \ell$, and $f \in L_1(\Omega)$ with $|f|_{W^\ell_p(\Omega^\gamma_T)} < \infty$,

$$\|D^\alpha(f - Q(f))\|_{L_p(T)} \le K h_T^{\ell-|\alpha|} |f|_{W^\ell_p(\Omega^\gamma_T)}, \tag{3.6}$$

where Ω^γ_T is the union of $\operatorname{star}^{2\gamma-1}(v)$ for all vertices v of T, and K depends only on n, p, d, $\omega_{\mathcal{T}}$, γ, $C := C_1C_2$, and the Lipschitz constant $L_{\partial\Omega}$ of the boundary $\partial\Omega$ of Ω. As a consequence, if $f \in W^\ell_p(\Omega)$, then for all $1 \le p \le \infty$ and $0 \le |\alpha| \le \ell - 1$,

$$\|D^\alpha(f - Q(f))\|^{\mathcal{T}}_{L_p(\Omega)} \le K' |\mathcal{T}|^{\ell-|\alpha|} |f|_{W^\ell_p(\Omega)}, \tag{3.7}$$

where K' depends only on n, p, d, $\omega_{\mathcal{T}}$, C, γ, and $L_{\partial\Omega}$.

Proof. Let us define the operator Q. In view of (3.4), each functional λ_k, $k = 1, \ldots, m$, is well-defined on $S|_{E_k}$. By Hahn-Banach theorem, we extend λ_k from $S|_{E_k}$ to $S_d(\mathcal{T})|_{E_k}$, such that

$$|\lambda_k s| \le C_2 \|s\|_{L_\infty(E_k)}, \quad \text{for all } s \in S_d(\mathcal{T})|_{E_k},$$

and define $\hat{Q} : S_d(\mathcal{T}) \to S$ by

$$\hat{Q}(s) = \sum_{k=1}^m \lambda_k(s|_{E_k}) s_k, \quad s \in S_d(\mathcal{T}).$$

Let T be a simplex in $\mathcal{T}$, and let $T \subset E_k$. In view of (3.1), $\operatorname{diam}(E_k) \le c_1 h_T$ and $\operatorname{vol}(E_k) \le c_2 \operatorname{vol}(T)$, with some constants c_1, c_2 depending only on n, γ and $\omega_{\mathcal{T}}$. (This is easy to show considering that the number of simplices in E_k is bounded by a constant depending only on n, γ and $\omega_{\mathcal{T}}$, and that any two simplices T', T'' with a common facet satisfy $h_{T''} \le \tilde{c}_1 h_{T'}$ and $\mathrm{vol}\,(T'') \le \tilde{c}_2 \mathrm{vol}\,(T')$, for some $\tilde{c}_1, \tilde{c}_2$ depending only on n and $\omega_{\mathcal{T}}$.) Hence, for any $s \in S_d(\mathcal{T})$ we have in view of (2.2),

$$c_3^{-1} \mathrm{vol}^{1/p}\,(T) \|s\|_{L_\infty(E_k)} \le \|s\|_{L_p(E_k)} \le c_4 \mathrm{vol}^{1/p}\,(T) \|s\|_{L_\infty(E_k)},$$

where c_3, c_4 depend only on $n, d, p, \gamma, \omega_{\mathcal{T}}$. Since the basis splines s_k are also piecewise polynomials, we have by (3.3)

$$\|s_k\|_{L_p(T)} \le c_5 \mathrm{vol}^{1/p}\,(T) \|s_k\|_{L_\infty(T)} \le c_5 \mathrm{vol}^{1/p}\,(T) C_1,$$

where c_5 depends only on n, d and p. Hence, assuming $s \in S_d(\mathcal{T})$ and applying (3.2) and (3.4), we obtain for any $T \in \mathcal{T}$,

$$\begin{aligned}
\|\hat{Q}(s)\|_{L_p(T)} &= \| \sum_{\substack{k=1\\ T \subset E_k}}^{m} \lambda_k(s|_{E_k}) s_k \|_{L_p(T)} \le \sum_{\substack{k=1\\ T \subset E_k}}^{m} |\lambda_k(s|_{E_k})| \|s_k\|_{L_p(T)} \\
&\le c_5 C_1 C_2 \sum_{\substack{k=1\\ T \subset E_k}}^{m} \|s\|_{L_\infty(E_k)} \mathrm{vol}^{1/p}\,(T) \\
&\le c_3 c_5 C_1 C_2 \sum_{\substack{k=1\\ T \subset E_k}}^{m} \|s\|_{L_p(E_k)} \le c_6 c_3 c_5 C_1 C_2 \|s\|_{L_p(\Omega_T^\gamma)},
\end{aligned}$$

where the last inequality follows from

$$\bigcup_{\substack{k=1\\ T \subset E_k}}^{m} E_k \subset \Omega_T^\gamma, \qquad T \in \mathcal{T},$$

and

$$\begin{aligned}
\#\{k : T \subset E_k\} &\le \dim S_d(\mathcal{T})|_{\Omega_T^\gamma} = \tbinom{n+d}{n} \#\{T' \in \mathcal{T} : T' \subset \Omega_T^\gamma\} \\
&\le \tbinom{n+d}{n} c_6,
\end{aligned}$$

where c_6 depends only on γ and $\omega_{\mathcal{T}}$. Thus, we have shown that

$$\|\hat{Q}(s)\|_{L_p(T)} \le c \|s\|_{L_p(\Omega_T^\gamma)}, \qquad s \in S_d(\mathcal{T}), \quad T \in \mathcal{T}, \tag{3.8}$$

with c depending only on n, p, d, $\omega_{\mathcal{T}}$, C, and γ.

In order to extend $Q = \hat{Q}$ from $S_d(\mathcal{T})$ to $L_1(\Omega)$, we consider, for any $f \in L_1(\Omega)$ and any $T \in \mathcal{T}$, the *average Taylor polynomial* [4] $p_T(f)$ of degree $\ell - 1$ with respect to the inscribed ball of T. By the Bramble-Hilbert lemma [4, p. 100],

$$|f - p_T(f)|_{W_p^k(T)} \le B h_T^{\ell-k} |f|_{W_p^\ell(T)}, \qquad 0 \le k \le \ell - 1, \tag{3.9}$$

where B depends only on $\ell, n, \omega_{\mathcal{T}}$. We define $\hat{s}(f) \in S_d(\mathcal{T})$ by

$$\hat{s}(f)|_T = p_T(f), \qquad T \in \mathcal{T},$$

and set

$$Q(f) := \hat{Q}(\hat{s}(f)), \qquad f \in L_1(\Omega).$$

Clearly, Q is a projector onto S, and, in particular, in view of (3.5),

$$Q(p) = p, \qquad \text{for any } p \in \Pi^n_{\ell-1}. \tag{3.10}$$

Let us prove (3.6). Suppose that $T \in \mathcal{T}$, $0 \le |\alpha| \le \ell - 1$, and $|f|_{W^\ell_p(\Omega^\gamma_T)} < \infty$. By using the Stein extension theorem (see e.g. [4, Theorem 1.4.5]), we extend $f|_{\Omega^\gamma_T}$ to a function $\tilde{f}$ defined on the convex hull U of Ω^γ_T such that

$$|\tilde{f}|_{W^\ell_p(U)} \le c_7 |f|_{W^\ell_p(\Omega^\gamma_T)},$$

where c_7 depends only on $n, p, \ell, \omega_{\mathcal{T}}$ and, possibly, on the Lipschitz constant $L_{\partial\Omega}$ if the boundary of Ω^γ_T contains a part of $\partial\Omega$. Now, let $q \in \Pi^n_{\ell-1}$ be the average Taylor polynomial for $\tilde{f}$ with respect to a ball in U of the greatest diameter. Again by the Bramble-Hilbert lemma, we have

$$|\tilde{f} - q|_{W^k_p(U)} \le c h_T^{\ell-k} |\tilde{f}|_{W^\ell_p(U)}, \qquad 0 \le k \le \ell - 1,$$

which implies, in particular,

$$\|D^\beta(f - q)\|_{L_p(\Omega^\gamma_T)} \le c_8 h_T^{\ell-|\beta|} |f|_{W^\ell_p(\Omega^\gamma_T)}, \qquad 0 \le |\beta| \le \ell - 1, \tag{3.11}$$

with c_8 depending only on $n, p, \ell, \omega_{\mathcal{T}}, L_{\partial\Omega}$. Therefore,

$$\begin{aligned}
\|D^\alpha(f - Q(f))\|_{L_p(T)} &\le \|D^\alpha(f - q)\|_{L_p(T)} + \|D^\alpha(q - Q(f))\|_{L_p(T)} \\
&\le c_8 h_T^{\ell-|\alpha|} |f|_{W^\ell_p(\Omega^\gamma_T)} + \|D^\alpha(q - Q(f))\|_{L_p(T)}.
\end{aligned}$$

By (3.10), (2.1), (3.8), and (3.11),

$$\begin{aligned}
\|D^\alpha(q - Q(f))\|_{L_p(T)} &= \|D^\alpha \hat{Q}(q - \hat{s}(f))\|_{L_p(T)} \\
&\le \frac{A}{h_T^{|\alpha|}} \|\hat{Q}(q - \hat{s}(f))\|_{L_p(T)} \le \frac{c}{h_T^{|\alpha|}} \|q - \hat{s}(f)\|_{L_p(\Omega^\gamma_T)} \\
&\le \frac{c}{h_T^{|\alpha|}} \|f - q\|_{L_p(\Omega^\gamma_T)} + \frac{c}{h_T^{|\alpha|}} \|f - \hat{s}(f)\|_{L_p(\Omega^\gamma_T)} \\
&\le c h_T^{\ell-|\alpha|} |f|_{W^\ell_p(\Omega^\gamma_T)} + \frac{c}{h_T^{|\alpha|}} \|f - \hat{s}(f)\|_{L_p(\Omega^\gamma_T)}.
\end{aligned}$$

Now, since $h_{T'} \le c_9 h_T$, for all $T' \in \mathcal{T}$ such that $T' \subset \Omega^\gamma_T$, where c_9 depends only on n, γ and $\omega_{\mathcal{T}}$, we have by (3.9)

$$\begin{aligned}
\|f - \hat{s}(f)\|_{L_\infty(\Omega^\gamma_T)} &= \max_{\substack{T' \in \mathcal{T} \\ T' \subset \Omega^\gamma_T}} \|f - p_{T'}(f)\|_{L_\infty(T')} \\
&\le B \max_{\substack{T' \in \mathcal{T} \\ T' \subset \Omega^\gamma_T}} h^\ell_{T'} |f|_{W^\ell_\infty(T')} \le B c_9 h^\ell_T |f|_{W^\ell_\infty(\Omega^\gamma_T)}
\end{aligned}$$

in the case $p = \infty$, and

$$\begin{aligned}
\|f - \hat{s}(f)\|^p_{L_p(\Omega^\gamma_T)} &= \sum_{\substack{T' \in \mathcal{T} \\ T' \subset \Omega^\gamma_T}} \|f - p_{T'}(f)\|^p_{L_p(T')} \\
&\le B^p \sum_{\substack{T' \in \mathcal{T} \\ T' \subset \Omega^\gamma_T}} h^{\ell p}_{T'} |f|^p_{W^\ell_p(T')} \le B^p c_9^p h^{\ell p}_T |f|^p_{W^\ell_p(\Omega^\gamma_T)}
\end{aligned}$$

in the case $1 \le p < \infty$, which completes the proof of (3.6).

To show (3.7), we first consider the case $p = \infty$. For some $T^* \in \mathcal{T}$, we have by (3.6),

$$\begin{aligned}\|D^\alpha(f - Q(f))\|^{\mathcal{T}}_{L_\infty(\Omega)} &= \|D^\alpha(f - Q(f))\|_{L_\infty(T^*)} \le K h_{T^*}^{\ell-|\alpha|} |f|_{W^\ell_\infty(\Omega^\gamma_{T^*})} \\ &\le K|\mathcal{T}|^{\ell-|\alpha|} |f|_{W^\ell_\infty(\Omega)}.\end{aligned}$$

Assume now that $1 \le p < \infty$. Then by (3.6),

$$\begin{aligned}\Big(\|D^\alpha(f - Q(f))\|^{\mathcal{T}}_{L_p(\Omega)}\Big)^p &= \sum_{T\in\mathcal{T}} \|D^\alpha(f - Q(f))\|^p_{L_p(T)} \\ &\le K^p \sum_{T\in\mathcal{T}} h_T^{(\ell-|\alpha|)p} |f|^p_{W^\ell_p(\Omega^\gamma_T)} \\ &\le K^p |\mathcal{T}|^{(\ell-|\alpha|)p} \sum_{T\in\mathcal{T}} \sum_{\substack{T'\in\mathcal{T}\\ T'\subset\Omega^\gamma_T}} |f|^p_{W^\ell_p(T')}.\end{aligned}$$

Now

$$\sum_{T\in\mathcal{T}} \sum_{\substack{T'\in\mathcal{T}\\ T'\subset\Omega^\gamma_T}} |f|^p_{W^\ell_p(T')} = \sum_{T\in\mathcal{T}} \#\{T' \in \mathcal{T} : T \subset \Omega^\gamma_{T'}\} |f|^p_{W^\ell_p(T)},$$

and since $T \subset \Omega^\gamma_{T'} \Leftrightarrow T' \subset \Omega^\gamma_T$, we have

$$\#\{T' \in \mathcal{T} : T \subset \Omega^\gamma_{T'}\} = \#\{T' \in \mathcal{T} : T' \subset \Omega^\gamma_T\} \le c_6,$$

and, hence,

$$\begin{aligned}\Big(\|D^\alpha(f - Q(f))\|^{\mathcal{T}}_{L_p(\Omega)}\Big)^p &\le c_6 K^p |\mathcal{T}|^{(\ell-|\alpha|)p} \sum_{T\in\mathcal{T}} |f|^p_{W^\ell_p(T)} \\ &= c_6 K^p |\mathcal{T}|^{(\ell-|\alpha|)p} |f|^p_{W^\ell_p(\Omega)},\end{aligned}$$

which completes the proof of (3.7). ■

Bases with properties (3.1)–(3.4) are called *stable local bases*. Indeed, (3.1) and (3.2) imply that the basis functions s_k have local support, and it follows from (3.1)–(3.4) that they are *stable* in L_∞ in the sense that for any real $\alpha_1, \dots, \alpha_m$,

$$K_1 \max_{1\le k\le m} |\alpha_k| \le \Big\| \sum_{k=1}^m \alpha_k s_k \Big\|_{L_\infty(\Omega)} \le K_2 \max_{1\le k\le m} |\alpha_k|, \tag{3.12}$$

where K_1, K_2 depend only on $n, d, \omega_{\mathcal{T}}, \gamma, C_1, C_2$.

It can also be shown [7, Lemma 6.2] that, after renorming, the basis $s_1, \dots, s_m$ is stable in L_p, $1 \le p < \infty$, that is

$$K_1 \Big(\sum_{k=1}^m |\alpha_k|^p\Big)^{1/p} \le \Big\| \sum_{k=1}^m \alpha_k \tilde{s}_k \Big\|_{L_p(\Omega)} \le K_2 \Big(\sum_{k=1}^m |\alpha_k|^p\Big)^{1/p}, \tag{3.13}$$

where

$$\tilde{s}_k = \mathrm{vol}^{-1/p}(E_k)\, s_k,$$

and K_1, K_2 depend only on $n, d, \omega_{\mathcal{T}}, \gamma, C_1, C_2$.

4. Spaces of smooth piecewise polynomials with stable local bases. Bases with properties (3.1)–(3.4), with $\gamma = 1$ and C_1, C_2 depending only on d, are available for $S_d^r(\mathcal{T})$ and certain subspaces of it on arbitrary $\mathcal{T}$ if $d \geq r2^n + 1$, see [7] and references therein. In the case of two variables, [10] provides a construction of bases for $S_d^r(\mathcal{T})$ satisfying (3.1)–(3.5), with $\gamma = 3$ and C_1, C_2 depending only on d, as soon as $d \geq 3r + 2$. There are many more results on stable local spline bases in recent literature, especially in the contexts of high order macro-elements and Lagrange interpolation methods, see e.g. [1, 2, 11, 15] and references therein. (See also [13].)

Some classical spaces of smooth finite elements, for example those based on Argyris element [4, Example 3.2.10], can be interpreted as so-called *super spline* subspaces of $S_d^r(\mathcal{T})$ [16], and their associated bases are easily seen to satisfy the requirements of Theorem 3.1.

Specifically, for the Argyris element we have in the notation of Theorem 3.1,

$$S = \{s \in S_5^1(\mathcal{T}) : \, s \text{ is } C^2 \text{ smooth at any vertex } v \text{ of } \mathcal{T}\}. \tag{4.1}$$

Clearly, (3.5) is satisfied with $\ell = 6$. The functionals $\lambda_k : S \to \mathbb{R}$ are function evaluations, weighted first and second derivatives at the vertices,

$$s(v),\ h_T \frac{\partial s}{\partial x_1}(v),\ h_T \frac{\partial s}{\partial x_2}(v),\ h_T^2 \frac{\partial^2 s}{\partial x_1^2}(v),\ h_T^2 \frac{\partial^2 s}{\partial x_2^2}(v),\ h_T^2 \frac{\partial^2 s}{\partial x_1 \partial x_2}(v), \tag{4.2}$$

and weighted first order normal derivatives at the middle points of the edges of $\mathcal{T}$,

$$h_T \frac{\partial s}{\partial n}\left(\tfrac{v_1+v_2}{2}\right), \tag{4.3}$$

where h_T is the diameter of a triangle in $\mathcal{T}$ containing the corresponding evaluation point v or $\frac{v_1+v_2}{2}$ from (4.2), (4.3). The sets $E_k = \operatorname{supp} s_k$ are either $\operatorname{star}(v)$ for the functionals of type (4.2), or the unions of two triangles sharing the edge $[v_1, v_2]$ in case (4.3). Hence, (3.1) and (3.2) are satisfied with $\gamma = 1$. Furthermore, by estimating the norms of local Hermite interpolation operators, it can be shown that $\|s_k\|_{L_\infty(\Omega)} \leq C_1$, where C_1 is an absolute constant, see [7, Lemma 3.3]. Similarly, in view of the inverse estimates (2.1), $|\lambda_k s| \leq C_2 \|s\|_{L_\infty(E_k)}$, for any $s \in S$, where C_2 is again an absolute constant, see [7, p. 292]. Thus, (3.3) and (3.4) hold true with $C = C_1 C_2$ being an absolute constant.

5. Stable Splitting $S = S_0 + S_b$. In the finite element method an important role is played by spaces of finite elements vanishing on (parts of) the boundary. We set

$$S_0 = \{s \in S : \, s|_{\partial\Omega} = 0\}.$$

Because of its utility in [3], we now consider the possibility to split S into a direct sum

$$S = S_0 + S_b$$

such that there exists a basis $\{s_1, \ldots, s_m\}$ for S satisfying the hypotheses (3.1)–(3.4) of Theorem 3.1 (i.e., a stable local basis), with $\{s_1, \ldots, s_{m_0}\}$ being a basis for S_0 and $\{s_{m_0+1}, \ldots, s_m\}$ a basis for S_b.

We provide a construction in the case $n = 2$ and $S \subset S_5^1(\Omega)$. Instead of (4.1), consider

$$S = \{s \in S_5^1(\mathcal{T}) : \, s \text{ is } C^2 \text{ smooth at any } \textit{interior} \text{ vertex } v \text{ of } \mathcal{T}\}. \tag{5.1}$$

Thus, in contrast to Argyris element, the functions in S are not necessarily C^2 at the boundary vertices.

We now describe a set of functionals $\{\lambda_1, \ldots, \lambda_m\} \subset S^*$, such that the desired basis $\{s_1, \ldots, s_m\}$ for S will be uniquely defined by duality

$$\lambda_i s_j = \delta_{i,j} := \begin{cases} 1, & \text{if } i = j, \\ 0, & \text{otherwise.} \end{cases}$$

The set $\{\lambda_1, \ldots, \lambda_m\}$ includes

(a) the functionals (4.2) for all *interior* vertices v of $\mathcal{T}$,

(b) the functionals (4.3) for all edges of $\mathcal{T}$, and

(c) the following functionals for each boundary vertex v of $\mathcal{T}$:

$$s(v),\ h_T \frac{\partial s}{\partial e_0}(v),\ h_T \frac{\partial s}{\partial e_0^\perp}(v),\ h_T^2 \frac{\partial^2 s}{\partial e_0^2}(v), \ldots,\ h_T^2 \frac{\partial^2 s}{\partial e_n^2}(v),\ h_T^2 \frac{\partial^2 s}{\partial e_0 \partial e_1}(v),$$

where $e_0, \ldots, e_n$ are all edges of $\mathcal{T}$ emanating from v, in counterclockwise order, with e_0 and e_n being the boundary edges.

Here the symbol $\frac{\partial}{\partial e}$ denotes the usual directional derivative in the direction of edge e, and $\frac{\partial}{\partial e^\perp}$ in the orthogonal direction. The above second order edge derivatives

$$\frac{\partial^2 s}{\partial e_0^2}(v), \ldots, \frac{\partial^2 s}{\partial e_n^2}(v), \frac{\partial^2 s}{\partial e_0 \partial e_1}(v)$$

are well defined (and independent from each other even if some edges are collinear) despite s being only C^1 at the boundary vertices, see [9]. This choice of the degrees of freedom at boundary vertices in (c) is motivated by the Morgan-Scott basis construction [14] and is shown to be stable in [9].

Following the argumentation in [9], one can see that the basis $\{s_1, \ldots, s_m\}$ for S, defined by duality, satisfies (3.1)–(3.4) with $\gamma = 1$ and bounded C_1, C_2. Hence it is a stable local basis. Moreover, (3.5) is obviously true for S with $n = 2$ and $\ell = 6$. Therefore, Theorem 3.1 applies to this basis.

To determine the subsets of $\{s_1, \ldots, s_m\}$ which generate S_0 and S_b, respectively, we now split the functionals in (c) into two groups (c1) and (c2) as follows.

(c1) The first group includes

$$h_T^2 \frac{\partial^2 s}{\partial e_1^2}(v), \ldots,\ h_T^2 \frac{\partial^2 s}{\partial e_{n-1}^2}(v),\ h_T^2 \frac{\partial^2 s}{\partial e_0 \partial e_1}(v),$$

for all boundary vertices, and, in addition, $h_T \frac{\partial s}{\partial e_0^\perp}(v)$ for those boundary vertices, where e_0 and e_n are collinear.

(c2) The second group includes

$$s(v),\ h_T \frac{\partial s}{\partial e_0}(v),\ h_T^2 \frac{\partial^2 s}{\partial e_0^2}(v),\ h_T^2 \frac{\partial^2 s}{\partial e_n^2}(v),$$

for all boundary vertices, and, in addition, $h_T \frac{\partial s}{\partial e_0^\perp}(v)$ for those boundary vertices, where e_0 and e_n are *not* collinear.

Let now $\{\lambda_1, \dots, \lambda_{m_0}\}$ list all functionals λ_i in (a), (b), and (c1), and let $\{\lambda_{m_0+1}, \dots, \lambda_m\}$ be those in (c2). It is easy to see that

$$S_0 = \{s \in S : \lambda_{m_0+1} s = \cdots = \lambda_m s = 0\}.$$

Therefore $S_0 = \operatorname{span}\{s_1, \dots, s_{m_0}\}$, and $S_b := \operatorname{span}\{s_{m_0+1}, \dots, s_m\}$ is its complement in S as required.

Clearly, both $\{s_1, \dots, s_{m_0}\}$ and $\{s_{m_0+1}, \dots, s_m\}$ are *stable local bases* as subsets of the stable local basis $\{s_1, \dots, s_m\}$.

Note that $h_T \dfrac{\partial s}{\partial e_0^\perp}(v)$ belongs to (c1) or (c2) depending on whether e_0 and e_n are *exactly* collinear or not. In particular it is in (c2) if e_0 and e_n are near-collinear, but not collinear, a situation which may appear quite often when a polygonal domain is an approximation of a smooth domain. As e_0 and e_n become exactly collinear, $h_T \dfrac{\partial s}{\partial e_0^\perp}(v)$ moves to (c1). Thus, the dimensions of S_0 and S_b jump if a vertex shared by collinear boundary edges is slightly perturbed. This 'dimension instability' is related to a well-known similar phenomenon in the theory of bivariate splines, where the dimension formulas for the spline spaces depend on some geometric information about the placement of the vertices. This behaviour is compatible with the availability of stable bases, see for example the discussion in [10, Remark 13.1].

Since λ_i are function evaluations or derivatives of at most second order, one may apply them to any sufficiently smooth functions, thus leading to the *interpolation operator* $I : C^2(\Omega) \to S$, defined by

$$I(f) = \sum_{i=1}^{m} \lambda_i(f) s_i. \tag{5.2}$$

An obvious property of this operator is

$$f|_{\partial\Omega} = 0 \qquad \Longrightarrow \qquad I(f) \in S_0. \tag{5.3}$$

It is easy to see that operator I can be used in the proof of Theorem 3.1 in place of Q, if one is only interested in sufficiently smooth f.

Finally, consider the operator $I_b : C^2(\Omega) \to S_b$, defined by

$$I_b(f) = \sum_{i=m_0+1}^{m} \lambda_i(f) s_i. \tag{5.4}$$

Clearly,

$$I_b(f)|_{\partial\Omega} = I(f)|_{\partial\Omega}. \tag{5.5}$$

Hence, by the trace theorem (see [4, Theorem 1.6.6]), we obtain for any $1 \le p \le \infty$,

$$\|f - I_b(f)\|_{L_p(\partial\Omega)} \le C \|f - I(f)\|_{L_p(\Omega)}^{1-1/p} \|f - I(f)\|_{W_p^1(\Omega)}^{1/p},$$

where C is a constant depending only on p and the Lipschitz constant $L_{\partial\Omega}$ of $\partial\Omega$. This allows to find a bound for $\|f - I_b(f)\|_{L_p(\partial\Omega)}$, see (5.8) below, using the estimates available for $\|f - I(f)\|_{L_p(\Omega)}$ and $\|D^\alpha(f - I(f))\|_{L_p(\Omega)}$, $|\alpha| = 1$.

We summarize the results proven in this section in the following theorem.

THEOREM 5.1. *For the space S defined in (5.1), there exists a basis $\{s_1, \dots, s_m\}$ satisfying (3.1)–(3.5), such that $\{s_1, \dots, s_{m_0}\}$ is a basis for $S_0 = \{s \in S : s|_{\partial\Omega} = 0\}$. Hence, the operator $Q : L_1(\Omega) \to S$ defined in the proof of Theorem 3.1 satisfies (3.6) and (3.7). Moreover, the interpolation operators $I : C^2(\Omega) \to S$ and $I_b : C^2(\Omega) \to S_b$ defined by (5.2) and (5.4), respectively, have the following properties.*

(a) If $f|_{\partial\Omega} = 0$, then $I(f) \in S_0$.

(b) Suppose that $f \in C^2(\Omega)$ and let $1 \le p \le \infty$, $\ell \le 6$, be such that $W_p^\ell(\Omega) \subset C^2(\Omega)$. Then for any $0 \le |\alpha| \le \ell - 1$,

$$\|D^\alpha(f - I(f))\|_{L_p(T)} \le K h_T^{\ell-|\alpha|} |f|_{W_p^\ell(\Omega_T^1)}, \quad \text{for any } T \in \mathcal{T}, \tag{5.6}$$

and, as a consequence

$$\|D^\alpha(f - I(f))\|_{L_p(\Omega)}^{\mathcal{T}} \le K' |\mathcal{T}|^{\ell-|\alpha|} |f|_{W_p^\ell(\Omega)}. \tag{5.7}$$

Moreover,

$$\|f - I_b(f)\|_{L_p(\partial\Omega)} \le K'' |\mathcal{T}|^{\ell-1/p} |f|_{W_p^\ell(\Omega)}. \tag{5.8}$$

The constants K, K', K'' depend only on p, $\omega_{\mathcal{T}}$, and the Lipschitz constant $L_{\partial\Omega}$.

REMARK 5.2. We conjecture that for any n, r with $d \ge r2^n + 1$, the techniques of [7] can be used to generalize Theorem 5.1 to $S = S_d^r(\mathcal{T})$ and certain subspaces of $S_d^r(\mathcal{T})$ defined similar to (5.1). In addition, an extension to bivariate piecewise polynomials on domains enclosed by picewise algebraic curves is under consideration.

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