On the Computation of Stable Local Bases for Bivariate Polynomial Splines

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Dedicated to Professor L. L. Schumaker on the occasion of his 60th birthday

Abstract. We show that stable local bases for the spaces of polynomial splines on a triangulation of a bivariate polygonal domain can be efficiently computed by using either singular value decomposition or pivoted QR-decompositon of certain small matrices of nodal smoothness conditions.

§1. Introduction

Let $\mathcal{S}_d^r(\triangle)$ denote the space of polynomial splines of degree d and smoothness r on a triangulation $\triangle$ of a polygonal domain Ω,

$$\mathcal{S}_d^r(\triangle) = \{s \in C^r(\Omega) : \ s|_T \in \mathcal{P}_d \ \text{ for all triangles } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of bivariate polynomials of total degree d. The question of constructing well-behaved bases for $\mathcal{S}_d^r(\triangle)$ is practically important, especially because the spaces $\mathcal{S}_d^r(\triangle_0), \mathcal{S}_d^r(\triangle_1), \ldots, \mathcal{S}_d^r(\triangle_n), \ldots$ are *nested* if the sequence of triangulations $\triangle_0, \triangle_1, \ldots, \triangle_n, \ldots$ of Ω is obtained by consecutive refinements of $\triangle_0$, see [5,13]. For the multiresolution applications a construction of a *stable locally supported basis* $s_1, \ldots, s_D$ for $\mathcal{S}_d^r(\triangle)$ is needed, where (L_p-) stability means that for all choices of the coefficient vector $c = (c_1, \ldots, c_D)$,

$$K_1 \|c\|_p \ \le \ \|\sum_{i=1}^{D} c_i s_i\|_p \ \le \ K_2 \|c\|_p, \tag{1}$$

with constants K_1, K_2 depending only on r, d and the smallest angle $\theta_\triangle$ in $\triangle$. Bases for $\mathcal{S}_d^r(\triangle)$ with this properties are known if $d \ge 3r+2$, see [8,9]. Standard finite-element bases available for $d \ge 4r+1$ (see *e.g.* [16]) are also stable and locally supported. However, they span subspaces of $\mathcal{S}_d^r(\triangle)$ that are

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not nested. The same applies to the stable local superspline bases constructed for $d \geq 3r+2$ in [3,12].

It is well-known [1,10,11,14,15] that the dimension of $\mathcal{S}_d^r(\triangle)$ depends on the geometry of the triangulation and is generally *instable*, *i.e.*, it may change as certain vertices are slightly perturbed. In other words, the number of basis functions suddenly changes. This may raise a question whether the stable local bases are practically computable in the presence of such anomalies.

The main purpose of this paper is to show that the answer to this question is positive for $r = 1$ and $r = 2$, which are the most important cases in practice. Note that we study in this respect the algorithm of constructing stable local basis by using orthogonal decomposition of matrices of nodal smoothness conditions suggested recently in [6]. This algorithm is easy to implement by using standard codes of computing singular value decomposition or pivoted QR-decomposition of small matrices, see Section 3. Moreover, it applies to spline spaces in more than two variables if the degree is high enough. However, in the case of bivariate splines we have to restrict ourselves to $d \geq 4r+1$ since this method does not work if $3r+2 \leq d \leq 4r$, in contrast to the more sophisticated methods of [9]. Obviously, this makes no difference if $r = 1$, and we have to consider $d \geq 9$ instead of $d \geq 8$ if $r = 2$.

The paper is organized as follows. In Section 2 we recall the construction of [6] in the case of splines of two variables. In Section 3 we discuss two orthogonal decompositions to be used to compute the bases for the null spaces of local smoothness matrices. Section 4 is devoted to the crucial question of the influence of the tolerance ε needed for the orthogonal decompositions because of the roundoff errors. Finally, in Section 5 we present the results of our numerical experiments showing that stable local basis splines with desired properties can indeed be efficiently computed.

§2. Construction of a Stable Local Basis

Let $\mathcal{V}$ and $\mathcal{E}$ be the sets of all vertices and all edges of the triangulation $\triangle$. Given an edge $e \in \mathcal{E}$, we denote by D_e and $D_{e^\perp}$ the derivative in the direction parallel or perpendicular to e, respectively. The linear functional evaluating any function at $\xi \in \Omega$ will be denoted by δ_ξ.

Consider the following set $\mathcal{N}$ of nodal linear functionals on $\mathcal{S}_d^r(\triangle)$,

$$\mathcal{N} = \Big(\bigcup_{v\in\mathcal{V}} \bigcup_{q=0}^{2r} \mathcal{N}_{v,q}\Big) \cup \Big(\bigcup_{e\in\mathcal{E}} \mathcal{N}_e\Big) \cup \Big(\bigcup_{T\in\triangle} \mathcal{N}_T\Big),$$

where for each $T = \langle v_1, v_2, v_3\rangle \in \triangle$,

$$\mathcal{N}_T = \{\delta_\xi : \ \xi \in \Xi_T\},$$

$$\Xi_T = \{\xi = \frac{i_1 v_1 + i_2 v_2 + i_3 v_3}{d} : \ i_1 + i_2 + i_3 = d, \quad i_1, i_2, i_3 > r\},$$

for each edge $e = \langle v_1, v_2\rangle$,

$$\mathcal{N}_e = \{\delta_\xi D_{e^\perp}^q : \ q = 0, \ldots, r, \quad \xi \in \Xi_{e,q}\},$$

$$\Xi_{e,q} = \{\xi = \frac{i_1 v_1 + i_2 v_2}{d} : \ i_1 + i_2 = d, \quad i_1, i_2 > 2r - q\},$$

and for each vertex $v \in \mathcal{V}$ and $q = 0, \ldots, 2r$, the set $\mathcal{N}_{v,q}$ is defined as follows. Let $T^{[i]} = \langle v, v_i, v_{i+1} \rangle$, $i = 1, \ldots, n_v$, be the triangles in $\triangle$ attached to v in counterclockwise order. (If v is an interior vertex, then $v_{n_v+1} = v_1$. If v lies on the boundary of Ω, then v_1 and v_{n_v+1} are distinct boundary vertices.) Let, furthermore, $e_i = \langle v, v_i \rangle$, $i = 1, \ldots, n_v + 1$, and $\theta_i = \angle e_i e_{i+1}$, $i = 1, \ldots, n_v$. We set

$$\mathcal{N}_{v,q} = \bigcup_{i=1}^{n_v} \{\delta_v D_{e_i}^{q-\alpha} D_{e_{i+1}}^{\alpha} \;:\; \alpha = 0, \ldots, q\},$$

where for any $s \in \mathcal{S}_d^r(\triangle)$, $D_{e_i}^{q-\alpha} D_{e_{i+1}}^{\alpha} s := D_{e_i}^{q-\alpha} D_{e_{i+1}}^{\alpha} s|_{T^{[i]}}$ if $q > r$. Since $\delta_v D_{e_i}^q s|_{T^{[i-1]}} = \delta_v D_{e_i}^q s|_{T^{[i]}}$ for all q, each functional $\delta_v D_{e_i}^q$ is present only once in $\mathcal{N}_{v,q}$.

Suppose that $d \geq 4r + 1$. A *complete* system of linear relations for the functionals in $\mathcal{N}$ (so called nodal smoothness conditions) is given by the equations

$$\sin^{-\alpha}\theta_i \, \delta_v D_{e_i}^{q-\alpha} D_{e_{i+1}}^{\alpha} - \sum_{\beta=0}^{\alpha} (-1)^{\beta} \binom{\alpha}{\beta} \mu_i^{\alpha-\beta} \sin^{-\beta}\theta_{i-1} \, \delta_v D_{e_{i-1}}^{\beta} D_{e_i}^{q-\beta} = 0, \quad (2)$$

for all $v \in \mathcal{V}$, $q = 1, \ldots, 2r$, $\alpha = 1, \ldots, \min\{r, q\}$, and for each i such that e_i is an *interior* edge (*i.e.*, $i = 1, \ldots, n_v$ if v is an interior vertex, and $i = 2, \ldots, n_v$ if v is a boundary vertex of $\triangle$). Here

$$\mu_i := \frac{\sin(\theta_{i-1} + \theta_i)}{\sin\theta_i \sin\theta_{i-1}}, \quad (3)$$

and we identify $v_{\ell+n_v} = v_\ell$, $e_{\ell+n_v} = e_\ell$ if v is an interior vertex. Note that nodal smoothness conditions were used in [4,6–8,14]. The complete system of linear relations (2) has been discussed in [8] for $r = 1$ and in [6] for the general case of splines of $n \geq 2$ variables.

Denoting by R the matrix of nodal smoothness conditions (2), we see that R is *block diagonal*, namely

$$R = [\, R_{\mathcal{V}} \quad O \,],$$

where

$$R_{\mathcal{V}} = \mathrm{diag}(R_v)_{v \in \mathcal{V}}, \qquad R_v = \mathrm{diag}(R_{v,q})_{q=1}^{2r},$$

each block $R_{v,q}$ corresponding to the relations between derivatives of order q at vertex v, and O is a zero matrix that corresponds to nodal functionals in $\mathcal{N}_{v,0}$, $v \in \mathcal{V}$, $\mathcal{N}_e$, $e \in \mathcal{E}$, and $\mathcal{N}_T$, $T \in \triangle$, not involved in any smoothness conditions.

Given any spline $s \in \mathcal{S}_d^r(\triangle)$, the column vector

$$\varphi(s) := (\nu s)_{\nu \in \mathcal{N}} \quad (4)$$

belongs to the null space of R,

$$\mathrm{null}(R) := \{a \in \mathbb{R}^{\#\mathcal{N}} \;:\; Ra = 0\}.$$

Moreover, the mapping $\varphi : \mathcal{S}_d^r(\triangle) \to \mathrm{null}(R)$ defined by (4) is a *linear isomorphism*, which follows from the facts that R is the matrix of a complete system of linear relations for $\mathcal{N}$, and that for each $T \in \triangle$, the functionals in

$$\mathcal{N}(T) := \Big(\bigcup_{v\in\mathcal{V}\cap T} \bigcup_{q=0}^{2r} \mathcal{N}_{v,q} \Big) \cup \Big(\bigcup_{\substack{e\in\mathcal{E}\\ e\subset T}} \mathcal{N}_e \Big) \cup \mathcal{N}_T$$

constitute a well-posed Hermite interpolation scheme for bivarite polynomials of total degree d, see *e.g.* [16]. Therefore, φ is invertible, and, given $a = (a_\nu)_{\nu\in\mathcal{N}} \in \mathrm{null}(R)$, each polynomial piece $s|_T$ of $s = \varphi^{-1}(a) \in \mathcal{S}_d^r(\triangle)$ can be computed from the interpolation conditions $\nu s|_T = a_\nu$, $\nu \in \mathcal{N}(T)$.

We will call every matrix A whose columns form a basis for $\mathrm{null}(R)$ a basis matrix for $\mathcal{S}_d^r(\triangle)$. Owing to the block diagonal structure of R, we can construct a basis matrix for $\mathcal{S}_d^r(\triangle)$ from blocks that are basis matrices with respect to the blocks of R. More precisely, for each $v \in \mathcal{V}$, $q = 1, \dots, 2r$, let $A_{v,q}$ be a matrix whose columns form a basis for $\mathrm{null}(R_{v,q})$. Let, furthermore,

$$A_{\mathcal{V}} := \mathrm{diag}(A_v)_{v\in\mathcal{V}}, \qquad A_v := \mathrm{diag}(A_{v,q})_{q=1}^{2r},$$

and I a unit matrix of size $\#\mathcal{V} + \#\mathcal{N}_e + \#\mathcal{N}_T$. Then

$$A = \begin{bmatrix} A_{\mathcal{V}} & O \\ O & I \end{bmatrix}$$

is obviously a basis matrix for $\mathcal{S}_d^r(\triangle)$. The number D of columns $a^{[1]}, \dots, a^{[D]}$ of A equals the dimension of $\mathcal{S}_d^r(\triangle)$, and the splines

$$s_i = \varphi^{-1}(a^{[i]}), \qquad i = 1, \dots, D, \tag{5}$$

form a basis for $\mathcal{S}_d^r(\triangle)$.

The following theorem is the restriction to splines of two variables of the main result of [6].

Theorem 1. *Let $d \geq 4r + 1$. Suppose that for each $v \in \mathcal{V}$ and $q = 1, \dots, 2r$, the columns of $A_{v,q}$ form an orthonormal basis for* $\mathrm{null}(R_{v,q})$*. Then the splines s_i defined by (5) form an L_p-stable local basis for $\mathcal{S}_d^r(\triangle)$, $1 \leq p \leq \infty$, after a proper renorming.*

We note that the support of each s_i is either a subset of the union of all triangles attached to a vertex, the union of at most two triangles sharing an edge, or a single triangle, depending on whether the i-th column of A corresponds to a vertex, an edge, or a triangle of $\triangle$.

Remark. The idea to analyse the null space of the matrix of smoothness conditions was suggested in [2] in the context of Bernstein-Bézier representation of bivariate splines. The advantage of nodal smoothness conditions used here is that the associated matrix R is block diagonal, even in the case of more than two variables, see [6].

§3. Orthogonal Decomposition of Smoothness Matrices

In order to comlete the algorithm of constructing stable local bases for $\mathcal{S}_d^r(\triangle)$, it remains to specify how we compute the basis matrices $A_{v,q}$. Two standard techniques from computational linear algebra to find an orthonormal basis for the null space of a matrix are based on the *singular value decomposition* and *pivoted QR-decomposition*, respectively, see *e.g.* [17].

SVD. Compute the singular value decomposition (SVD) of $R_{v,q}$,

$$R_{v,q} = Q_L S Q_R^T,$$

where Q_L, Q_R are orthogonal matrices, S is of the same size as $R_{v,q}$,

$$S = \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & \sigma_m & 0 & \cdots & 0 \end{bmatrix},$$

$\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_{\rho_{svd}} > \varepsilon \geq \sigma_{\rho_{svd}+1} \geq \cdots \geq \sigma_m \geq 0$ are the *singular values* of $R_{v,q}$, with $\varepsilon > 0$ being a tolerance. Then $A_{v,q}$ is obtained from Q_R by removing its first ρ_{svd} columns.

PQR. Compute the pivoted QR-decomposition (PQR) of $R_{v,q}^T$, $R_{v,q}^T P^T = QS^T$, *i.e.*,

$$PR_{v,q} = SQ^T,$$

where P is a permutation matrix, Q is an orthogonal matrix, S is a lower triangular matrix of the same size as $R_{v,q}$,

$$S = \begin{bmatrix} d_1 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \times & d_2 & \cdots & 0 & \vdots & & \vdots \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ \times & \times & \cdots & d_m & 0 & \cdots & 0 \end{bmatrix},$$

the diagonal elements of S satisfy $|d_1| \geq |d_2| \geq \cdots \geq |d_{\rho_{pqr}}| > \varepsilon \geq |d_{\rho_{pqr}+1}| \geq \cdots \geq |d_m| \geq 0$, with $\varepsilon > 0$ being a tolerance. Then $A_{v,q}$ is obtained from Q by removing its first ρ_{pqr} columns.

Note that in theory we can simply take $\varepsilon = 0$, such that $\rho_{svd} = \rho_{pqr} = \operatorname{rank}(R_{v,q})$. However, for practical computations it is necessary to introduce the tolerance ε because of the unavoidable roundoff errors in the entries of $R_{v,q}$. As we will see in the next sections, the choice of ε is decisive for the performance of the algorithm.

§4. Choice of Tolerance

A well known difficulty in the implementation of the bases for the full spline spaces $\mathcal{S}_d^r(\triangle)$ is the fact that the dimension of $\mathcal{S}_d^r(\triangle)$ is instable in general, *i.e.*, a small perturbation of the location of vertices of a triangulation may lead to a change in the dimension of the space, see *e.g.* [15]. In other words, in the presence of some exceptional geometric configurations (such as a quadrilateral *singular cell* where the central vertex is the intersection point of two diagonals) the dimension of $\mathcal{S}_d^r(\triangle)$ is higher than generically. This means that after a small perturbation of an exceptional configuration, say a perturbation due to the roundoff, the number of basis functions drops. For example, the dimension of the space $S_5^1(\triangle)$ on a singular cell $\triangle$ is 44, whereas it is only 43 on a near-singular cell, see Fig. 1.

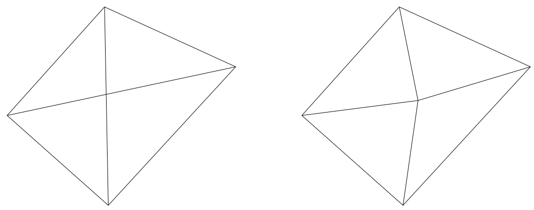

Fig. 1. Singular, respectively near-singular cell.

In order to figure out what actually happens when we use the above algorithm to compute a basis for the spline space on such a *near-exceptional* configuration, we first make the following *observations*.

- The piecewise polynomials s_i (more precisely, their *local degrees of freedom* such as Bézier coefficients of each $s_i|_T$) can always be computed from (5) by using the Hermite interpolation scheme specified for each $T \subset \operatorname{supp} s_i$ by the set of nodal functionals $\mathcal{N}(T)$. This computation only requires to solve for each T a system of $\binom{d+2}{2}$ linear equations that is well-conditioned provided the smallest angle $\theta_\triangle$ of $\triangle$ is not too small.
- Regardless of the geometry of the triangulation and the choice of the tolerance ε, s_i are always *continuous* and have *local supports.* They satisfy the *stability* requirement (1) with constants depending only on r, d and $\theta_\triangle$, since the columns of A are orthonormal.
- If the tolerance ε is too small, it may happen that we rule out some basis functions that are needed to span $\mathcal{S}_d^r(\triangle)$.
- If the tolerance ε is too high, we may get some additional basis functions that are actually not in the space $\mathcal{S}_d^r(\triangle)$.

Let us consider this last case in more detail. Given $\varepsilon > 0$, let

$$A^\varepsilon = [a^{[1]} \cdots a^{[D^\varepsilon]}] = \begin{bmatrix} A_{\mathcal{V}}^\varepsilon & O \\ O & I \end{bmatrix}$$

be the basis matrix computed by the above algorithm using either SVD or PQR decomposition of $R_{v,q}$'s with tolerance ε. The splines

$$s_i = \varphi^{-1}(a^{[i]}), \qquad i = 1, \dots, D^\varepsilon,$$

are certainly in $\mathcal{S}_d^0(\triangle)$. Moreover, s_i is C^r-smooth and as such lies in $\mathcal{S}_d^r(\triangle)$ if and only if $Ra^{[i]} = 0$. Suppose the first M^ε columns of A^ε include elements of the submatrix $A^\varepsilon_{\mathcal{V}}$. Then the splines s_i, $i = M^\varepsilon + 1, \ldots, D^\varepsilon$, are always in $\mathcal{S}_d^r(\triangle)$. Each s_i, $i = 1, \ldots, M^\varepsilon$, corresponds to a block $A^\varepsilon_{v_i,q_i}$ of $A^\varepsilon_{\mathcal{V}}$ and is C^r-smooth if and only if $R_{v_i,q_i}\tilde{a}^{[i]} = 0$, where $\tilde{a}^{[i]}$ is the column of $A^\varepsilon_{v_i,q_i}$ consisting of the components of a_i. Thus, that our additional splines are not in $\mathcal{S}_d^r(\triangle)$ simply means that $R_{v_i,q_i}\tilde{a}^{[i]}$ is *nonzero.*

However, due to the roundoff errors, $R_{v_i,q_i}\tilde{a}^{[i]}$ *can never equal zero exactly* on a computer, except some trivial situations, even if s_i is theoretically in $\mathcal{S}_d^r(\triangle)$. Therefore, a spline s_i should be regarded numerically C^r-smooth as soon as $\|R_{v_i,q_i}\tilde{a}^{[i]}\|_\infty$ is reasonably small.

Fortunately, we can easily estimate $\|R_{v_i,q_i}\tilde{a}^{[i]}\|_\infty$ in terms of the tolerance ε. Indeed, for each s_i, $i = 1, \ldots, M^\varepsilon$, we have

$$\|R_{v_i,q_i}\tilde{a}^{[i]}\|_2 \le \sigma_{\rho_{svd}+1} \le \varepsilon \qquad \text{for SVD,}$$
$$\|R_{v_i,q_i}\tilde{a}^{[i]}\|_\infty \le |d_{\rho_{pqr}+1}| \le \varepsilon \qquad \text{for PQR,}$$

where ρ_{svd}, ρ_{pqr} are chosen as in Section 3. The first inequality is obvious, and for a proof of the second one see *e.g.* [17, p. 370].

Thus, we conclude that *the splines* $s_1, \ldots, s_{D^\varepsilon}$ *are numerically* C^r*-smooth if the tolerance* ε *is reasonably small.* On the other hand, since we do not want to miss any basis splines needed to span $\mathcal{S}_d^r(\triangle)$, *the tolerance should not be too small.*

It is a well known fact that the singular values continuously depend on the matrix, and hence they do not change significantly if the matrix $R_{v,q}$ is slightly perturbed (say, by the roundoff errors). Therefore, if we are using SVD to compute the matrices $A^\varepsilon_{v,q}$ and if the accuracy of the computations is high enough (*i.e.*, the *rounding unit* ε_M is sufficiently small), then we can choose the tolerance ε as small as we want without missing the essential basis splines. The PQR decomposition is cheaper to compute, but it is not as reliable as SVD and there are examples when it fails to reveal the (numerical) rank deficiency of a matrix, see [17, p. 374]. This means that we do risk to miss some basis functions if we are using PQR. (Note that this has never happened in our numerical experiments described below.)

§5. Numerical Tests: Cells with Four Edges

The disscussion in the previous section shows that it is crucial to find minimal values of tolerance ε for which the above algorithm produces enough basis functions to span the full space $\mathcal{S}_d^r(\triangle)$. To get an idea how small this ε can be, we performed a number of numerical experiments for C^r splines, $r = 1, 2$. According to Section 2, an orthogonal decomposition has to be computed for smoothness matrices $R_{v,q}$, $q = 1, \ldots, 2r$ for each vertex v. We restrict ourselves to *interior vertices with four attached edges*, since this includes the most interesting cases of singular and near-singular cells as in Fig. 1. It follows from well-known formulas for the dimension of spline spaces on cells [15], that for C^1 splines,

$$\dim \mathrm{null}(R_{v,1}) = 2,$$
$$\dim \mathrm{null}(R_{v,2}) = \begin{cases} 5, & \text{if } v \text{ is singular,} \\ 4, & \text{otherwise,} \end{cases}$$

and for C^2 splines,

$$\dim \mathrm{null}(R_{v,1}) = 2, \qquad \dim \mathrm{null}(R_{v,2}) = 3,$$
$$\dim \mathrm{null}(R_{v,3}) = \begin{cases} 6, & \text{if } v \text{ is singular,} \\ 5, & \text{if } v \text{ is semisingular,} \\ 4, & \text{otherwise,} \end{cases}$$
$$\dim \mathrm{null}(R_{v,4}) = \begin{cases} 9, & \text{if } v \text{ is singular,} \\ 8, & \text{otherwise,} \end{cases}$$

where semisingular means that two of the edges attached to v are parallell, and the other two are not.

Note that the matrix $R_{v,1}$ is the same for both $r = 1$ and $r = 2$, and that in C^1 case we investigate the simplified matrix

$$\tilde{R}_{v,2} = \begin{bmatrix} -\mu_1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & -\mu_2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -\mu_3 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & -\mu_4 & 1 \end{bmatrix}$$

instead of

$$R_{v,2} = \begin{bmatrix} -\mu_1 & \sin^{-1}\theta_1 & 0 & 0 & 0 & 0 & 0 & \sin^{-1}\theta_4 \\ 0 & \sin^{-1}\theta_1 & -\mu_2 & \sin^{-1}\theta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sin^{-1}\theta_2 & -\mu_3 & \sin^{-1}\theta_3 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \sin^{-1}\theta_3 & -\mu_4 & \sin^{-1}\theta_4 \end{bmatrix}.$$

(See Section 2 for the definitions of μ_i and θ_i.)

Our experiments were performed with MATLAB (version 5.3) on a Sun Ultra 60 workstation. At least 50 millions cells with four edges have been tested, with the smallest angle always at least $\pi/10$. The results are presented in Tab. 1 (for SVD) and Tab. 2 (for PQR decomposition).

r	matrix	size	ε	t_{mean}	δ_{max}	θ_{max}
1 or 2	$R_{v,1}$	4×4	$14\varepsilon_M$	1.3939e-04	3.9968e-15	—
1	$\tilde{R}_{v,2}$	4×8	0	2.2251e-04	3.6082e-15	2.9086e-16
2	$R_{v,2}$	8×8	$68\varepsilon_M$	3.7429e-04	2.3981e-14	—
2	$R_{v,3}$	8×12	$24\varepsilon_M$	5.7217e-04	3.1974e-14	6.4575e-15
2	$R_{v,4}$	8×16	$28\varepsilon_M$	6.5097e-04	2.4869e-14	5.6034e-15

Tab. 1. SVD.

r	matrix	size	ε	t_{mean}	δ_{max}	θ_{max}
1 or 2	$R_{v,1}$	4×4	$22\varepsilon_M$	1.3051e-04	4.2188e-15	—
1	$\tilde{R}_{v,2}$	4×8	$1.8\varepsilon_M$	1.7471e-04	4.1356e-15	4.6638e-16
2	$R_{v,2}$	8×8	$51\varepsilon_M$	2.7389e-04	3.0198e-14	—
2	$R_{v,3}$	8×12	$27\varepsilon_M$	3.5814e-04	2.9976e-14	3.8858e-15
2	$R_{v,4}$	8×16	$60\varepsilon_M$	4.4264e-04	2.9643e-14	7.5763e-15

Tab. 2. PQR.

Here ε is the tolerance ($\varepsilon_M = 2^{-52} \approx 2.2204\text{e-}16$ denotes the rounding unit in the standard double-precision arithmetic) chosen such that the number of columns of $A^{\varepsilon}_{v,q}$ constructed with the corresponding decomposition is greater or equal $\dim \operatorname{null}(R_{v,q})$. The last three colunms of each table measure the efficiency of the algorithm: t_{mean} is the average CPU time in seconds used to compute SVD, respectively PQR decomposition of a matrix (note that we are using the MATLAB built-in functions `svd` and `qr`), $\delta_{\max}$ is the maximal value of $\|R_{v,q}a\|_\infty$ for the columns a of all matrices $A^{\varepsilon}_{v,q}$ appeared in the tests, and $\theta_{\max}$ is the maximum of the angles between $\langle v, v_i \rangle$ and $\langle v_{i+2}, v\rangle$ in all cases when the algorithm considers the edges $e_i = \langle v, v_i\rangle$ and $e_{i+2} = \langle v, v_{i+2}\rangle$ to be parallell, which leads to constructing additional basis functions. Since the dimensions of $\operatorname{null}(R_{v,1})$ and (in C^2 case) $\operatorname{null}(R_{v,2})$ are stable, $\theta_{\max}$ does not apply to these matrices.

These experiments show that the basis functions for $\mathcal{S}^r_d(\triangle)$, $r = 1, 2$, on a cell with four edges can be computed by using resonably small tolerance ε. The basis functions are numerically C^r-smooth and additional basis functions only appear if the cell is "numerically singular", respectively "numerically semisingular".

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