Characterization of the Best Uniform Approximation of Periodic Functions by Convex Classes Defined by Strictly CVD Kernels

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Abstract. We present a Chebyshev-type characterization for the best uniform approximations of periodic continuous functions by functions of the class

$$\mathcal{M} = \{f(x): \ f(x) = \int_0^{2\pi} K(x,y)h(y)\,dy, \ |h(y)| \le 1 \text{ a. e.}, \ y \in [0,2\pi)\},$$

where $K(x,y)$ is a strictly cyclic variation diminishing kernel.

§1 Introduction

Let $W^r_\infty[a,b]$ be the Sobolev class of functions in $C^{r-1}[a,b]$ whose $(r-1)$st derivative is absolutely continuous and whose rth derivative is an element of the unit ball of $L^\infty[a,b]$ and let W^r_∞ be the analogous class of 2π-periodic functions.

N. Korneichuk obtained in 1961 (see [4], p. 225) a characterization for the best approximation of a continuous 2π-periodic function by functions of the class W^1_∞. His characterization was evidently valid in the non-periodic case for the class $W^1_\infty[a,b]$.

In 1980 U. Sattes [6] extended this result to all the classes $W^r_\infty[a,b]$ with $r \ge 2$. It turned out that the best approximation coincides in some subinterval with a perfect spline of degree r satisfying some Chebyshev-type conditions.

Shortly thereafter A. Pinkus [5], motivated by Sattes' result, considered the class

$$\mathcal{M}_{[0,1]} = \{f(x): \ f(x) = \int_0^1 K(x,y)h(y)\,dy, \ l(y) \le h(y) \le u(y)\},$$

where $u, l \in C[0,1]$, fixed, $u > l$, and where $K(x,y)$ is a continuous strictly totally positive kernel. For this class he obtained existence, uniqueness and characterization of the best approximation to $f \in C[0,1]$ from $\mathcal{M}_{[0,1]}$, with the best approximation being a generalized perfect spline.

It is the aim of the paper to present a periodic analog of Pinkus' result.

Let $C(\mathbf{T})$ be the space of all 2π-periodic continuous functions f, under the Chebyshev norm $\|f\| = \max\{|f(x)| : \ x \in \mathbf{R}\}$.

For a function $f \in C(\mathbf{T})$, let $Z_c(f)$ denote the number of zeros of f in a period and let $S_c^-(f)$ and $S_c^+(f)$ denote the two usual measures of the number of cyclic sign changes of f (see [3], p. 250–261 for definitions), $S_c^-(f) \le Z_c(f) \le S_c^+(f)$. A 2π-periodic Borel measure μ is said to have $2n$ ($n = 1, 2, \dots$) relevant cyclic sign changes, denoted by $S_c(\mu)$, if there exist disjoint sets $A_1 < \dots < A_{2n} < A_1 + 2\pi$, with $\bigcup_{i=1}^{2n} A_i = [a, a+2\pi)$ for some real a, such that $(-1)^i \mu$ is a nonnegative measure on A_i and $\mu(A_i) \ne 0$, $i = 1, \dots, 2n$. If f is a summable 2π-periodic function, then by $S_c^-(f)$ we mean $S_c(\mu)$, where $f(y)\,dy = d\mu(y)$.

Definition 1.1. We say that a kernel $K(x,y) \in C(\mathbf{T}^2)$ is *strictly cyclic variation diminishing*, abbreviated $SCVD$, if for any nonzero 2π-periodic Borel measure μ we have $S_c^+(f) \le S_c(\mu)$, where $f(x) = \int_0^{2\pi} K(x,y)\,d\mu(y)$.

It follows from the definition that for every set of points $y_1 < \dots < y_{2k-1} < y_1 + 2\pi$ the functions $K(x, y_1), \dots, K(x, y_{2k-1})$ form a Chebyshev system. In view of this it is easy to prove that for each $SCVD$ kernel there exists a sequence of signs $\varepsilon_k = \pm 1$, $k = 1, 2, \dots$, such that

$$\varepsilon_k \det \|K(x_i, y_j)\|_{i,j=1}^{2k-1} > 0, \tag{1.1}$$

for any two sets of points $x_1 < \dots < x_{2k-1} < x_1 + 2\pi$ and $y_1 < \dots < y_{2k-1} < y_1 + 2\pi$, $k = 1, 2, \dots$.

Let us consider the class

$$\mathcal{M} = \{f(x) : \ f(x) = \int_0^{2\pi} K(x,y) h(y)\,dy, \ \ |h(y)| \le 1 \ \text{a. e.}, \ \ y \in [0, 2\pi)\},$$

where $K(x,y)$ is an $SCVD$ kernel.

Definition 1.2. A function $f \in \mathcal{M}$, $f(x) = \int_0^{2\pi} K(x,y) h(y)\,dy$, is said to be a *generalized perfect spline* relative to $\mathcal{M}$ if there exists a nonnegative integer $n = 0, 1, \dots$, such that

(I) If $n = 0$, then $h(y) = 1$ or -1 a. e.

(II) If $n \ge 1$, then there exist *knots* $\xi_1 < \dots < \xi_{2n} < \xi_1 + 2\pi$, such that $h(y) = (-1)^{j+1}$ a. e., $\xi_j < y < \xi_{j+1}$, $j = 1, \dots, 2n$, $\xi_{2n+1} = \xi_1 + 2\pi$.

Definition 1.3. A function $g \in C(\mathbf{T})$ is said to *equioscillate* on $2k$ points if there exist $\theta_1 < \dots < \theta_{2k} < \theta_1 + 2\pi$, such that $(-1)^i g(\theta_i) = \delta \|g\|$, $i = 1, \dots, 2k$, for some $\delta \in \{-1, 1\}$.

We can now formulate our main result.

Theorem 1.1. *Assume that $K(x,y)$ is SCVD and $g \in C(\mathbf{T}) \setminus \mathcal{M}$. The best approximation $f^* \in \mathcal{M}$,*

$$f^*(x) = \int_0^{2\pi} K(x,y)h^*(y)\,dy\,,$$

to g from $\mathcal{M}$ is uniquely characterized as follows.

There exists a nonnegative integer $n = 0,1,2,\ldots$ such that f^* is a generalized perfect spline relative to $\mathcal{M}$ with $2n$ knots $\xi_1 < \cdots < \xi_{2n} < \xi_1 + 2\pi$, such that

(I) *If $n = 0$, then there exists a real θ, for which $(g - f^*)(\theta) = \varepsilon_1 \operatorname{sign} h^* \|g - f^*\|$.*

(II) *If $n \geq 1$, then*

$$h^*(y) = (-1)^{j+1} \text{ a. e., } \ \xi_j < y < \xi_{j+1}, \ j = 1,\ldots,2n, \tag{1.2}$$

and at least one of the following two assertions is true.

(IIa) *$g - f^*$ equioscillates on $2n+2$ points.*

(IIb) *There exist $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$, such that*

$$\det \|K(\theta_i, \xi_j)\|_{i,j=1}^{2n} = 0 \tag{1.3}$$

and for some θ'_{2n}, θ''_{2n} satisfying $\theta_{2n-1} < \theta'_{2n} \leq \theta_{2n} \leq \theta''_{2n} < \theta_1 + 2\pi$,

$$\begin{aligned}(g - f^*)(\theta_1) = -(g - f^*)(\theta_2) = \cdots = (g - f^*)(\theta_{2n-1}) = \\ = -(g - f^*)(\theta'_{2n}) = -(g-f^*)(\theta''_{2n}) = \varepsilon_n \varepsilon \|g - f^*\|,\end{aligned} \tag{1.4}$$

where ε_n satisfies (1.1), and

$$\varepsilon = \operatorname{sign}\, \det \left\| \int_{\xi_j}^{\xi_{j+1}} K(\theta_i, y)\,dy \right\|_{i,j=1}^{2n}. \tag{1.5}$$

Remark . Theorem 1.1 remains true if we omit condition (IIa). The proof of this fact is rather long, and we do not include it in the paper.

The proof of Theorem 1.1 is given in Section 3. Section 2 presents some preliminaries.

§2 Preliminaries

Definition 2.1. A linear independent system of functions $u_1, \dots, u_{2n} \in C(\mathbf{T})$, where $n = 1, 2, \dots$, is said to be *quasi-Chebyshev*, abbreviated QT, if $u_1, \dots, u_{2n-1}$ is a Chebyshev system and $Z_c(f) \le 2n$ for any nontrivial polynomial $f = a_1 u_1 + \cdots + a_{2n} u_{2n}$. The $2n$-dimensional linear space spanned by a QT-system is called a *quasi-Chebyshev space* or a *QT-space.*

Quasi-Chebyshev systems were introduced in [1]. We need some of their properties.

Let U be the QT-space spanned by a QT-system $u_1, \dots, u_{2n} \in C(\mathbf{T})$. A set of $2n$ points $t_1 < \cdots < t_{2n} < t_1 + 2\pi$ is said to be an *I-set* relative to U if $\det \|u_i(t_j)\|_{i,j=1}^{2n} \neq 0$, and an *$NI$-set* otherwise.

It is easy to see that for any NI-set $\tau_1 < \cdots < \tau_{2n} < \tau_1 + 2\pi$ there exists a function $u_\tau \in U$, called a *winding polynomial*, such that $u_\tau(\tau_j) = 0$, $j = 1, \dots, 2n$ and $\varepsilon(-1)^j u(t) > 0$, $\tau_j < t < \tau_{j+1}$, with $\varepsilon = \pm 1$. For a given NI-set the winding polynomial is easily checked to be uniquely determined up to a nonzero real factor.

Lemma 2.1.[1] *Let $\tau_1 < \cdots < \tau_{2n} < \tau_1 + 2\pi$ be an NI-set and let the set $T = \{t_1, \dots, t_{2n}\}$ satisfy $\tau_j < t_j < \tau_{j+1}$, $j = 1, \dots, 2n$. Then T is an I-set .*

The following theorem is a reformulation of Theorem 1 in [1].

Theorem 2.1. *Let $u_1, \dots, u_{2n} \in C(\mathbf{T})$ be a QT-system and let U be the corresponding QT-space . Suppose that $f \in C(\mathbf{T}) \setminus U$. A polynomial $u \in U$ is a best approximation to f from U if and only if there exists an NI-set relative to U, $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$, such that*

$$\begin{aligned} (f-u)(\theta_1) = -(f-u)(\theta_2) = \cdots = (f-u)(\theta_{2n-1}) = \\ = -(f-u)(\theta'_{2n}) = -(f-u)(\theta''_{2n}) = \pm\|f-u\| \end{aligned} \tag{2.1}$$

for some θ'_{2n}, θ''_{2n} satisfying $\theta_{2n-1} < \theta'_{2n} \le \theta_{2n} \le \theta''_{2n} < \theta_1 + 2\pi$.

In the rest of this section we consider some properties of $SCVD$ kernels.

Proposition 2.1. *Let $K(x,y)$ be an $SCVD$ kernel and let $\gamma_1 < \cdots < \gamma_{2n} < \gamma_1 + 2\pi$, $n = 1, 2, \dots$ be a set of points. Then both systems of functions $K(x, \gamma_1), \dots, K(x, \gamma_{2n})$ and $K(\gamma_1, y), \dots, K(\gamma_{2n}, y)$ are quasi-Chebyshev.*

Proof: Add a point $\gamma_{2n+1} \in (\gamma_{2n}, \gamma_1 + 2\pi)$. The proposition follows from the fact that, by (1.1), all the systems $\{K(x, \gamma_1), \dots, K(x, \gamma_{2n-1})\}$,

$\{K(x,\gamma_1),\ldots,K(x,\gamma_{2n+1})\}$, $\{K(\gamma_1,y),\ldots,K(\gamma_{2n-1},y)\}$ and $\{K(\gamma_1,y),\ldots,K(\gamma_{2n+1},y)\}$ are Chebyshev. ■

Proposition 2.2. *Let two sets of points* $\xi_1 < \cdots < \xi_{2n} < \xi_1 + 2\pi$ *and* $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$ *satisfy (1.3) for a given SCVD kernel* $K(x,y)$.

Then for any $\{y_1,\ldots,y_{2n}\}$ *such that* $\xi_j < y_j < \xi_{j+1}$, $j = 1,\ldots,2n$, *we have*

$$\operatorname{sign} \det \|K(\theta_i,y_j)\|_{i,j=1}^{2n} = \varepsilon \neq 0, \tag{2.2}$$

where ε *satisfies (1.5).*

Proof: It follows from (1.3) that $\{\xi_1,\ldots,\xi_{2n}\}$ is an NI-set relative to QT-space U spanned by $\{K(\theta_1,y),\ldots,K(\theta_{2n},y)\}$. Therefore, by Lemma 2.1, $\{y_1,\ldots,y_{2n}\}$ is an I-set and the determinant on the left-hand side of (2.2) is nonzero. If there exist $y_j,\, y_j' \in (\xi_j,\xi_{j+1})$, $j = 1,\ldots,2n$, such that $\det \|K(\theta_i,y_j)\|_{i,j=1}^{2n} \det \|K(\theta_i,y_j')\|_{i,j=1}^{2n} < 0$, then $\{y_j^* = \alpha y_j + (1-\alpha) y_j'\}_{j=1}^{2n}$ is an NI-set for some $\alpha \in (0,1)$. This is impossible in view of Lemma 2.1. ■

Proposition 2.3 *Let* $K(x,y)$ *be an SCVD kernel, and let* μ *be a nonzero* 2π*-periodic Borel measure such that* $(-1)^{j+1}\mu$ *is a nonnegative measure on* (ξ_j,ξ_{j+1}), $j = 1,\ldots,2n$, *with* $\xi_1 < \cdots < \xi_{2n} < \xi_1 + 2\pi$.

Assume that $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$ *satisfy (1.3). Then there exist* $i = 1,\ldots,2n$, *for which* $\varepsilon_n \varepsilon (-1)^i f(\theta_i) < 0$, *where* $f(x) = \int_0^{2\pi} K(x,y)\,d\mu(y)$, ε_n *satisfies (1.1), and* ε *satisfies (1.5).*

Proof: Suppose, to the contrary, that

$$\varepsilon_n \varepsilon (-1)^i f(\theta_i) \geq 0, \quad i = 1,\ldots,2n. \tag{2.3}$$

Because of the $SCVD$-property of the kernel, we have $S_c(\mu) \geq S_c^+(f) \geq 2n$ and, therefore, μ is nonzero on each interval (ξ_j,ξ_{j+1}), $j = 1,\ldots,2n$. Consider the representation

$$f(\theta_i) = \sum_{j=1}^{2n} (-1)^{j+1} \varphi_j(\theta_i), \quad i = 1,\ldots,2n, \tag{2.4}$$

where $\varphi_j(x) = \int_{\xi_j}^{\xi_{j+1}} K(x,y)\,|d\mu(y)|$, $j = 1,\ldots,2n$. By Proposition 2.2,

$$\delta \stackrel{\text{def}}{=} \det \|\varphi_j(\theta_i)\|_{i,j=1}^{2n} =$$

$$= \int_{\xi_1}^{\xi_2} \cdots \int_{\xi_{2n}}^{\xi_1+2\pi} \det \|K(\theta_i,y_j)\|_{i,j=1}^{2n} \,|d\mu(y_{2n})| \cdots |d\mu(y_1)| \neq 0$$

and $\operatorname{sign}\delta = \varepsilon$. Hence, we can derive from (2.4) that

$$1 = \sum_{k=1}^{2n} (-1)^{k+1} f(\theta_k) \delta^{-1} \det \|\varphi_j(\theta_i)\|_{i,j=1,\, i\neq k,\, j\neq 1}^{2n}. \qquad (2.5)$$

However, in view of (1.1),

$$\operatorname{sign} \det \|\varphi_j(\theta_i)\|_{i,j=1,\, i\neq k,\, j\neq 1}^{2n} =$$
$$= \operatorname{sign} \int_{\xi_2}^{\xi_3} \cdots \int_{\xi_{2n}}^{\xi_1+2\pi} \det \|K(\theta_i, y_j)\|_{i,j=1,\, i\neq k,\, j\neq 1}^{2n} |d\mu(y_{2n})| \cdots |d\mu(y_2)| = \varepsilon_n .$$

Because of this it follows from (2.3) that each term on the right-hand side of (2.5) is nonpositive. The left-hand side cannot be 1. This contradiction proves the proposition. ■

§3 Proof of Theorem 1.1

It follows from the compactness of the class $\mathcal{M}$, that for any $g \in C(\mathbf{T})$ there exists an element of best approximation $f^* \in \mathcal{M}$. The same analysis as one used by K. Glashoff in [2], shows that f^* is a generalized perfect spline relative to $\mathcal{M}$, $f^*(x) = \int_0^{2\pi} K(x,y) h^*(y)\, dy$. Therefore, by the convexity of $\mathcal{M}$, f^* is unique (see the proof of Corollary 2 in [2]).

If f^* has no knots, then it is readily seen that f^* can be characterized by the condition (I) of Theorem 1.1.

Let f^* have $2n$ knots $\xi_1 < \cdots < \xi_{2n} < \xi_1 + 2\pi$, $n \geq 1$, so that (1.2) holds. We will show that f^* can be characterized by the condition (II) of Theorem 1.1.

Sufficiency. If we have $\|g - f\| < \|g - f^*\|$ for a function $f(x) = \int_0^{2\pi} K(x,y) h(y)\, dy \in \mathcal{M}$, then $S_c^+(f^* - f) \leq S_c^-(h^* - h) \leq 2n$. On the other hand, in the case (IIa), $S_c^+(f^* - f) = S_c^+((g-f) - (g-f^*)) \geq 2n+2$, a contradiction. In the case (IIb) there exist $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$ satisfying (1.3) such that (1.4) holds for some θ'_{2n}, θ''_{2n} with $\theta_{2n-1} < \theta'_{2n} \leq \theta_{2n} \leq \theta''_{2n} < \theta_1 + 2\pi$. From (1.4) it follows that

Necessity. We first prove the following assertion.

(A) *The zero function is a best approximation to $g - f^*$ from the QT-space U^* spanned by the QT-system $K(x, \xi_1), \ldots, K(x, \xi_{2n})$.*

Following the argument similar to that in the proof of Lemma 4.4 in [5], we suppose, to the contrary, that there exist $u = \sum_{j=1}^{2n} a_j K(x, \xi_j) \in U^*$

such that $\|g - f^* - u\| < \|g - f^*\|$. Then, for $\gamma \in (0,1)$, $\|g - f^* - \gamma u\| < \|g - f^*\| - \gamma c$, where $c = \|g - f^*\| - \|g - f^* - u\| > 0$. Set

$$f_\gamma(x) = 2\sum_{j=1}^{2n}(-1)^j \int_{\xi_j}^{\xi_j'} K(x,y)\,dy,$$

where $\xi_j' = \xi_j + (-1)^j(\gamma/2)a_j$, $j = 1,\dots,2n$. Then $f^* + f_\gamma \in \mathcal{M}$ provided γ is sufficiently small. On the other hand, by the continuity of $K(x,y)$, we have $\|\gamma u - f_\gamma\| = o(\gamma)$. Thus, $\|g-(f^*+f_\gamma)\| \le \|g-f^*-\gamma u\|+\|\gamma u-f_\gamma\| \le \|g - f^*\| - \gamma c + o(\gamma)$, which, since $c > 0$, contradicts the assumption that f^* is the best approximation to g from $\mathcal{M}$. Therefore (A) is true.

It follows from Theorem 2.1 that there exists an NI-set relative to U^*, $\theta_1 < \cdots < \theta_{2n} < \theta_1 + 2\pi$, such that $(g - f^*)(\theta_1) = -(g - f^*)(\theta_2) = \cdots = (g - f^*)(\theta_{2n-1}) = -(g - f^*)(\theta_{2n}') = -(g - f^*)(\theta_{2n}'') = \lambda\|g - f^*\|$ for some θ_{2n}', θ_{2n}'' satisfying $\theta_{2n-1} < \theta_{2n}' \le \theta_{2n} \le \theta_{2n}'' < \theta_1 + 2\pi$, and with $\lambda = +1$ or -1.

Assume that f^* does not satisfy condition (IIa) of Theorem 1.1. In order to check condition (IIb) it will be sufficient to prove that

$$\lambda = \varepsilon_n\varepsilon \ , \tag{3.1}$$

where ε_n and ε satisfy (1.1) and (1.5), respectively.

Because $g - f^*$ equioscillates on at most $2n$ points, there exist two sets of points, $\{t_i'\}_{i=1}^{2n}$ and $\{t_i''\}_{i=1}^{2n}$, $t_1' \le \theta_1 \le t_1'' < t_2' \le \theta_2 \le t_2'' < \cdots < t_{2n}' \le \theta_{2n}' \le \theta_{2n}'' \le t_{2n}'' < t_1' + 2\pi$, such that $(g - f^*)(t_i') = (g - f^*)(t_i'') = (-1)^{i+1}\|g - f^*\|$, $i = 1,\dots,2n$, $(-1)^i(g - f^*)(t) < \|g - f^*\|$, $t \in (t_{i-1}'', t_{i+1}')$, $i = 1,\dots,2n$. In each interval (t_i'', t_{i+1}') we now choose a point $\tau_i \in (t_i'', t_{i+1}')$. By Lemma 2.1, $\{\tau_i\}_{i=1}^{2n}$ is an I-set relative to U^*, so that $\det\|K(\tau_i,\xi_j)\|_{i,j=1}^{2n} \ne 0$. Transforming continuously ξ_j into ξ_{j+1}, $j = 1,\dots,2n$, we can find points $y_1,\dots,y_{2n}$, $y_j \in (\xi_j,\xi_{j+1})$, such that $\det\|K(\tau_i,y_j)\|_{i,j=1}^{2n} = 0$. Then $\{\tau_i\}_{i=1}^{2n}$ is an NI-set relative to the QT-space spanned by $\{K(x,y_1),\dots,K(x,y_{2n})\}$. Let $w_\tau(x) = \sum_{j=1}^{2n} a_j K(x,y_j)$ be the corresponding winding polynomial, with $a_1 = 1$. Therefore, $w_\tau(x) = \int_0^{2\pi} K(x,y)\,d\mu_0(y)$, where $\mu_0(y) = \sum_{j=1}^{2n} a_j\delta(y - y_j)$ and $\delta(y)$ is the Dirac measure. Since $S_c(\mu_0) \ge S_c^+(w_\tau) = 2n$, we have

$$\operatorname{sign} a_j = (-1)^{j+1}, \ \ j = 1,\dots,2n. \tag{3.2}$$

It follows from Proposition 2.3 and the definition of winding polynomial that $\varepsilon_n\varepsilon(-1)^i w_\tau(\theta_i) < 0$, $i = 1,\dots,2n$.

If (3.1) is not valid, *i.e.*, $\lambda = -\varepsilon_n\varepsilon$, then $\lambda(-1)^i w_\tau(t) > 0$, $t \in [t_i', t_i'']$. Therefore,

$$\|g - f^* + \gamma w_\tau\| < \|g - f^*\| \tag{3.3}$$

if $\gamma > 0$ is small enough. Set

$$w_\tau^h(x) = \sum_{j=1}^{2n} a_j \ (2h)^{-1} \int_{y_j - h}^{y_j + h} K(x,y)\, dy, \quad h > 0.$$

Then, by (1.2) and (3.2), $f^* - \gamma w_\tau^h \in \mathcal{M}$ when $\gamma > 0$ and $h > 0$ are sufficiently small. Furthermore, $\|w_\tau - w_\tau^h\| \to 0$ as $h \to 0$. Thus, in view of (3.3), $\|g - (f^* - \gamma w_\tau^h)\| < \|g - f^*\|$ for some γ and h, which contradicts the optimality of f^*. ■

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