Refinable C^2 Piecewise Quintic Polynomials on Powell-Sabin-12 Triangulations

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Abstract

We present a construction of nested spaces of C^2 macro-elements of degree 5 on triangulations of a polygonal domain obtained by uniform refinements of an initial triangulation and a Powell-Sabin-12 split.

1 Introduction

A sequence of piecewise polynomial spaces $S_0, S_1, \ldots, S_n, \ldots$ with respect to triangulations $\Delta_0, \Delta_1, \ldots, \Delta_n, \ldots$ of a domain $\Omega \subset \mathbb{R}^2$ is said to be *nested* if $S_n \subset S_{n+1}$ for all $n = 0, 1, \ldots$. Nested spaces of smooth piecewise polynomials (splines) are used in multilevel algorithms for surface compression [7, 9], nonlinear approximation [2, 4] and preconditioning of spline based finite element system matrices [1, 6, 10].

If the triangulations Δ_n are obtained by successive refinements of a starting triangulation Δ_0, then the spaces $S_n = S^r_d(\Delta_n)$ of all C^r splines of degree at most d are nested. However, these spaces are known to possess stable local bases important for application only if d is relatively large, $d \geq 3r + 2$, see [5]. Therefore much attention is paid to the *macro-element* spaces [8, Chapter 6] whose degree can be kept much lower at the expense of requiring that Δ_n is obtained from a general triangulation by splitting each triangle into subtriangles by various methods such as Clough-Tocher split, Powell-Sabin-6 or Powell-Sabin-12 split. Some C^1 macro-elements, such as piecewise quadratic Powell-Sabin-12 element or cubic Fraeijs de Veubeke-Sanders element, are *refinable* in the sense that nested spline spaces with stable local bases can be constructed with their help. However, refinable macro-elements of higher smoothness have not been known.

In this paper we propose the first construction of refinable C^2 macro-elements, whose degree 5 is substantially lower than degree 8 of the splines of [5] and degree 9 of the refinable C^2 spline spaces with stable dimension suggested in [3]. On a single macro-triangle our spaces coincide with the C^2 quintic macro-element of [11], although we

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obtain a simpler description of it (important for nestedness) in the case when the central point of the Powell-Sabin-12 split is placed at the barycentre of the macro-triangle. The nestedness of the spaces is achieved as in [3] by relaxing the C^3 smoothness conditions at the vertices of macro-triangles, which allows to break the 'super-smoothness disks' at the vertices into half-disks. The proposed macro-elements are easy to implement in the framework of the Bernstein-Bézier techniques because we provide explicit formulas for all B-coefficients which are not computed directly by the standard smoothness conditions.

The paper is organised as follows. In Section 2 we recall the basics of the Bernstein-Bézier techniques used throughout the paper. Sections 3 and 4 are devoted to the construction of nested spaces and stable minimal determining sets (leading to a stable local basis), as well as the proofs of the main results, whereas Section 5 provides a nodal minimal determining set and error bounds for the corresponding Hermite interpolation operator.

2 Bernstein-Bézier techniques

We recall basic notions of the Bernstein-Bézier techniques, see [8] for details. Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, any bivariate polynomial p of total degree d can be uniquely represented in the form

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^{T,d}, \tag{1}$$

where $B_{ijk}^{T,d}$ are the Bernstein basis polynomials of degree d associated with T. We refer to the representation (1) as the *B-form* of p related to T. The c_{ijk}'s are called the *B-coefficients* of p, and the associated set of *domain points* is defined by

$$\mathcal{D}_{d,T} := \left\{ \xi_{ijk} := \frac{iv_1 + jv_2 + kv_3}{d} \right\}_{i+j+k=d}. \tag{2}$$

Given a regular triangulation $\Delta = \{T_i\}_{i=1}^N$ of a bounded connected polygonal domain $\Omega \subseteq \mathbb{R}^2$ and a positive integer d, we define the corresponding set of domain points by

$$\mathcal{D}_{d,\Delta} := \bigcup_{T \in \Delta} D_{d,T}.$$

(Recall that a *regular* triangulation is such that the union of all triangles of Δ is $\bar{\Omega}$ and the intersection of any pair of triangles of Δ either consists of a common edge or a common vertex of both triangles or is empty.)

Let $S_d^0(\Delta)$ be the space of continuous splines of degree d on Δ,

$$S_d^0(\Delta) := \{s \in C^0(\Omega) : s|_{T_i} \in \mathbb{P}_d,\ i = 1, \dots, N\},$$

where $\mathbb{P}_d$ denotes the space of all bivariate polynomials of total degree at most d. Given $s \in S_d^0(\Delta)$ and $T \in \Delta$, there exists a unique set of coefficients $\{c_\xi\}_{\xi \in \mathcal{D}_{d,T}}$ such that

$$s|_T = \sum_{\xi \in \mathcal{D}_{d,T}} c_\xi B_\xi^{T,d},$$

and each spline in $S_d^0(\Delta)$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\Delta}}$.

Given $0 \le m \le d$ and $T := \langle v_1, v_2, v_3\rangle$, we say that a domain point ξ_{ijk} is at a *distance* $\mathrm{dist}(\xi, v_1) = d - i$ from the vertex v_1 and at a distance $\mathrm{dist}(\xi, e_1) = i$ from the edge $e_1 = \langle v_2, v_3\rangle$ opposite to v_1. Furthermore, we refer to the set of domain points $R_m^T(v_1) := \{\xi_{d-m,j,m-j}\}_{j=0}^m$ as the *ring* of radius m around the vertex v_1. We refer to the set $D_m^T(v_1) := \bigcup_{n=0}^m R_n^T(v_1)$ as the *disk* of radius m around the vertex v_1. The rings and disks around v_2 and v_3 are defined similarly. If v is a vertex of Δ with triangles $T_1, \dots, T_k$ attached to it, then the ring and the disk of radius m around v are defined by $R_m(v) = \bigcup_{i=1}^k R_m^{T_i}(v)$ and $D_m(v) = \bigcup_{i=1}^k D_m^{T_i}(v)$, respectively.

Suppose now that S is a linear subspace of $S_d^0(\Delta)$ defined by enforcing some set of smoothness conditions across the edges of the triangulation Δ. Then a *determining set* for S is a subset M of the set of domain points $\mathcal{D}_{d,\Delta}$ such that if we set the B-coefficients c_ξ of some spline $s \in S$ to zero for all $\xi \in M$, then $s \equiv 0$. If M is a determining set for a spline space S and M has the smallest cardinality among all possible determining sets for S, then we call M *a minimal determining set* (MDS) for S. It is known that M is a MDS for S if and only if every spline $s \in S$ is uniquely determined by its set of B-coefficients $\{c_\xi\}_{\xi\in M}$.

An MDS M is called *local* provided that there is an integer ℓ such that for every $\xi \in \mathcal{D}_{d,\Delta} \cap T$ and every $T \in \Delta$, the B-coefficient c_ξ of a spline $s \in S$ is a linear combination of $\{c_\eta\}_{\eta\in\Gamma_\xi}$ where Γ_ξ is a subset of M with $\Gamma_\xi \subset \mathrm{star}^\ell(T)$. Here $\mathrm{star}^\ell(T) := \mathrm{star}(\mathrm{star}^{\ell-1}(T))$ for $l \ge 2$, where if U is a cluster of triangles, $\mathrm{star}(U) = \mathrm{star}^1(U)$ is the set of all triangles which have a nonempty intersection with some triangle of U. Moreover, M is said to be *stable* provided that there is a constant K depending only on d and the smallest angle in Δ such that

$$|c_\xi| \le K \max_{\eta\in\Gamma_\xi} |c_\eta|, \qquad \text{for all } \xi \in \mathcal{D}_{d,\Delta}.$$

We say that a spline $s \in S_d^0(\Delta)$ is C^ρ smooth at the vertex v provided that all polynomials $s|_T$ such that T is a triangle with vertex at v have common partial derivatives up to order ρ at the point v. In this case we write $s \in C^\rho(v)$.

Smoothness across an edge is described with the help of smoothness functionals defined as follows. Let $T = \langle v_1, v_2, v_3\rangle$ and $\tilde{T} = \langle v_4, v_3, v_2\rangle$ be two adjoining triangles which share the edge $e = \langle v_2, v_3\rangle$, and let c_{ijk} and $\tilde{c}_{ijk}$ be the coefficients of the B-representations of s_T and $s_{\tilde{T}}$, respectively. Then for any $n \le m \le d$, let $\tau_{e,m}^n$ be the linear functional defined on $S_d^0(\Delta)$ by

$$\tau_{e,m}^n s = \tilde{c}_{n,m-n,d-m} - \sum_{i+j+k=n} c_{i,j+d-m,k+m-n} B_{ijk}^{T,n}(v_4). \tag{3}$$

In terms of these linear functionals, the condition that s be C^r smooth across the edge e is equivalent to

$$\tau_{e,m}^n s = 0, \quad n \le m \le d, \; 0 \le n \le r.$$

3 Refinable spaces of C^2 piecewise quintics

Let Ω be a bounded connected polygonal domain in $\mathbb{R}^2$. Suppose that some initial regular triangulation Δ_0 of Ω is given. Beginning with Δ_0 we construct a sequence

$\{\Delta_n\}_{n=0}^{\infty}$ of triangulations of Ω by *uniform refinement*, that is Δ_{n+1} is obtained from Δ_n by subdividing any triangle T of Δ_n into four equal subtriangles by joining the midpoints of the three edges with each other, as in Figure 1 (left). The uniform refinement of a single triangle T will be denoted T_U.

Starting from $\{\Delta_n\}_{n=0}^{\infty}$ we introduce further subdivisions by splitting each triangle of Δ_n into twelve triangles by joining the midpoints of the three edges with each other and with the opposite vertices. This *Powell-Sabin-12 split* T_{PS12} of a single triangle T is illustrated in Figure 1 (right). Clearly, T_{PS12} is a refinement of T_U. The triangulation obtained from Δ_n by applying the Powell-Sabin-12 split to each triangle will be denoted Δ_n^*. An important observation is that Δ_{n+1}^* is a *refinement* of Δ_n^*, in the sense that Δ_{n+1}^* can be obtained from Δ_n^* by subdividing its triangles.

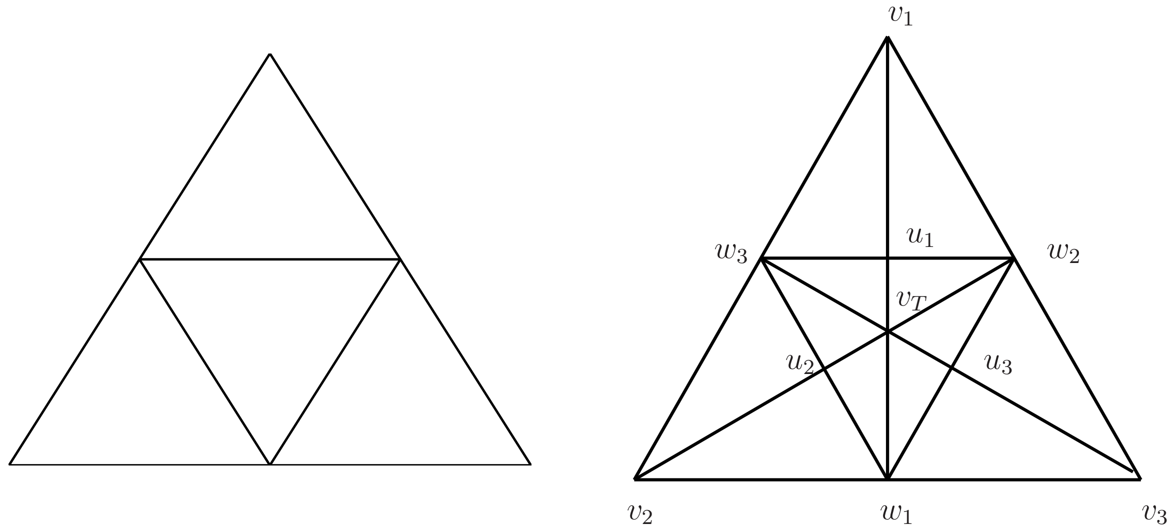

Figure 1: Uniform refinement T_U and Powell-Sabin-12 split T_{PS12} of a triangle.

For each $n = 0, 1, \ldots,$ we denote by $\mathcal{V}_n$, $\mathcal{E}_n$, $\tilde{\mathcal{E}}_n$ and $\mathcal{W}_n$ the sets of all vertices, edges, interior edges and midpoints of edges of Δ_n, respectively. Given a triangle $T \in \Delta_n$, we denote the vertices of T_{PS12} by $v_1, v_2, v_3, w_1, w_2, w_3, u_1, u_2, u_3$ and v_T as shown in Figure 1. We refer to the edges of the form $[v_i, u_i]$ as *type-1 edges*, to edges of the form $[u_i, v_T]$ as *type-2 edges* and to edges of the form $[w_i, v_T]$ as *type-3 edges*. For $i = 1, 2, 3$, we write $\mathcal{E}_n^i$ for the set of all edges of Δ_n^* of type-i.

We set

$$\tilde{\mathcal{V}}_0 = \emptyset, \quad \tilde{\mathcal{V}}_n = (\mathcal{V}_n \cap \operatorname{Int}\Omega) \setminus \mathcal{V}_0, \quad n = 1, 2, \ldots.$$

Then

$$\mathcal{V}_n \setminus \tilde{\mathcal{V}}_n = \mathcal{V}_0 \cup (\mathcal{V}_n \cap \partial\Omega).$$

For any $v \in \bigcup_{n\in\mathbb{N}} \tilde{\mathcal{V}}_n$, let $n_v := \min\{n : v \in \tilde{\mathcal{V}}_n\}$. Clearly, there is a unique edge e_v of Δ_{n_v-1}, with adjacent triangles $T_v^+, T_v^- \in \Delta_{n_v-1}$, such that v lies at the midpoint of e_v. Since $\mathcal{W}_n \cap \operatorname{Int}\Omega \subset \tilde{\mathcal{V}}_{n+1}$, the triangles $T_w^+, T_w^- \in \Delta_n$ are well defined for any $w \in \mathcal{W}_n \cap \operatorname{Int}\Omega$.

For $n = 0, 1, \ldots$, let $S_5^2(\Delta_n^*)$ denote the space of C^2 quintic piecewise polynomials, i.e.

$$S_5^2(\Delta_n^*) := \{s \in C^2(\Omega) : s|_T \in \mathbb{P}_5 \text{ for all } T \in \Delta_n^*\}.$$

We consider the subspace S_n of $S_5^2(\Delta_n^*)$ defined by

$$S_n = \big\{s \in S_5^2(\Delta_n^*) : \\ \text{(i) } s \in C^3(v) \quad \text{for all } v \in \mathcal{V}_0 \cup (\mathcal{V}_n \cap \partial\Omega) \text{ and all } v \in \mathcal{W}_n \cap \partial\Omega, \\ \text{(ii) } s|_{T_v^+} \in C^3(v), s|_{T_v^-} \in C^3(v) \quad \text{for all } v \in \tilde{\mathcal{V}}_n \text{ and all } v \in \mathcal{W}_n \cap \operatorname{Int}\Omega, \text{ and} \\ \text{(iii) } s \text{ is } C^3 \text{ across all edges in } \mathcal{E}_n^1 \cup \mathcal{E}_n^2 \cup \mathcal{E}_n^3\big\}.$$

One crucial property of the spaces S_n is their nestedness.

Theorem 1. *The spaces $S_n, n = 0, 1, \ldots$ are nested, that is,*

$$S_n \subset S_{n+1}, \quad n = 0, 1, 2, \ldots.$$

Proof. Let $n \geq 1$. We suppose $s \in S_{n-1}$ and show that $s \in S_n$. If $v \in \mathcal{V}_0 \cup (\mathcal{V}_n \cap \partial\Omega)$, then $v \in \mathcal{V}_0 \cup (\mathcal{V}_{n-1} \cap \partial\Omega)$ or $v \in \mathcal{W}_{n-1} \cap \partial\Omega$, so $s \in C^3(v)$ by Condition (i) in the definition of S_n. It is also clear that $s \in C^3(v)$ for $v \in \mathcal{W}_n \cap \partial\Omega$ since v lies in the interior of a boundary edge of Δ_{n-1}^*. If $v \in \tilde{\mathcal{V}}_n$, then either $v \in \tilde{\mathcal{V}}_{n-1}$, or $n_v = n$ and $v \in \mathcal{W}_{n-1} \cap \operatorname{Int}\Omega$, $T_v^+, T_v^- \in \Delta_{n-1}$ and v lies at the midpoint of the common edge e_v of these two triangles. By Condition (ii) in both cases $s|_{T_v^+}, s|_{T_v^-} \in C^3(v)$ as required. If $v \in \mathcal{W}_n \cap \operatorname{Int}\Omega$, then $n_v = n + 1$ and $T_v^+, T_v^- \in \Delta_n$, whereas v lies at the midpoint of the common edge $e_v \in \Delta_n$ of these two triangles. Moreover, for a triangle $T \in \Delta_{n-1}$, v is either the midpoint of the edges $\langle v_1, w_2\rangle, \langle v_1, w_3\rangle, \langle v_2, w_1\rangle, \langle v_2, w_3\rangle, \langle v_3, w_1\rangle, \langle v_3, w_2\rangle$ or the vertices u_1, u_2, u_3 of T_{PS12}. In the first case, $s|_{T_v^+}, s|_{T_v^-} \in C^3(v)$ because v is not a vertex of Δ_{n-1}^*. In the second case, $s|_{T_v^+}, s|_{T_v^-} \in C^3(v)$ since s is C^3 across type 1 and type 2 edges, $e \in \mathcal{E}_{n-1}^1 \cup \mathcal{E}_{n-1}^2$. If $e \in (\mathcal{E}_n^1 \cup \mathcal{E}_n^2 \cup \mathcal{E}_n^3)$ then either e is (a part of) an edge $\tilde{e} \in (\mathcal{E}_{n-1}^1 \cup \mathcal{E}_{n-1}^2 \cup \mathcal{E}_{n-1}^3)$ since Δ_n^* is a refinement of Δ_{n-1}^* or e lies in the interior of some triangle $T \in \Delta_{n-1}^*$. In both cases, s is C^3 across e by Condition (iii). $\square$

We now want to generate a stable local MDS for S_n.

For each $v \in \tilde{\mathcal{V}}_n$, let e_v in Δ_{n_v-1} be the unique edge with adjacent triangles $T_v^+, T_v^- \in \Delta_{n_v-1}$ such that v lies at the midpoint of e_v. For each $v \in \mathcal{V}_n$, we choose a triangle $\hat{T}_v \in \Delta_n^*$ with vertex at v. If $v \in \tilde{\mathcal{V}}_n$, we assume that $\hat{T}_v \subset T_v^+$ and we choose another triangle $\tilde{T}_v = \langle v, u, w\rangle \in \Delta_n^*$ attached to v such that $\tilde{T}_v \subset T_v^-$ and an edge of $\tilde{T}_v$ is a part of e_v. We now set $M_v := D_3(v) \cap \hat{T}_v$ for any $v \in \mathcal{V}_n$, and $\tilde{M}_v := M_v \cup \xi_{2,3,0}^{\tilde{T}_v}$ for any $v \in \tilde{\mathcal{V}}_n$. The set $\tilde{M}_v$ is illustrated in Figure 2.

Furthermore, for each edge e of Δ_n, let v_{T_e} be the barycentre of a triangle T_e in Δ_n attached to e, let w_e be the midpoint of e, let $T_e^3 = \langle v_{T_e}, w_e, u\rangle$ be one of the triangles in Δ_n^* attached to the edge $\langle w_e, v_{T_e}\rangle$, of type 3, and let $M_e = \big\{\xi_{3,2,0}^{T_e^3}, \xi_{2,3,0}^{T_e^3}, \xi_{2,2,1}^{T_e^3}, \xi_{1,4,0}^{T_e^3}\big\}$. For each $w \in \mathcal{W}_n \cap \operatorname{Int}\Omega$, let T_e^+ and T_e^- be two triangles in Δ_n attached to the edge $e = \langle v_1, v_2\rangle$ in Δ_n, such that w is the midpoint of e, that is $w = w_e$. Let $v_{T_e^+}$ be the barycentre of T_e^+ and let $\hat{T}_e^3 \subset T_e^+$ be some triangle in Δ_n^* attached to the edge $\langle w_e, v_{T_e^+}\rangle$ of type 3. Let $\tilde{T}_e = \langle w_e, u, v_1\rangle \subset T_e^-$ be one of the triangles in Δ_n^* with vertex w_e and such that one of its edges is a part of e. Let $\tilde{M}_e = \big\{\xi_{3,2,0}^{\hat{T}_e^3}, \xi_{2,3,0}^{\hat{T}_e^3}, \xi_{2,2,1}^{\hat{T}_e^3}, \xi_{1,4,0}^{\hat{T}_e^3}\big\} \cup \big\{\xi_{2,3,0}^{\tilde{T}_e}\big\}$. The domain points corresponding to $\tilde{M}_e$ are shown in Figure 3.

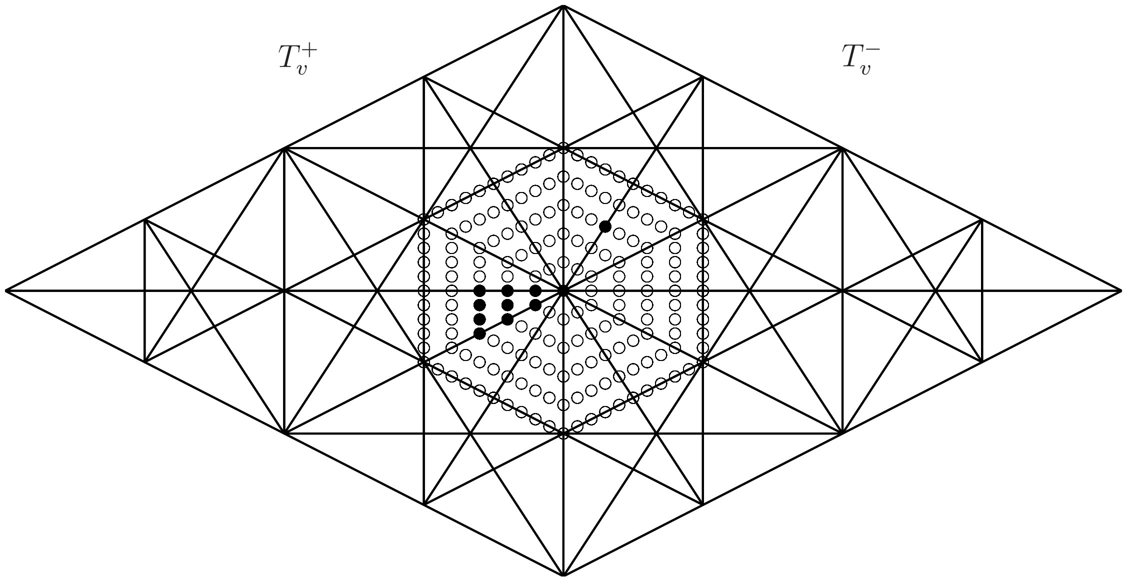

Figure 2: Domain points corresponding to $\tilde{M}_v$ are marked with filled circles.

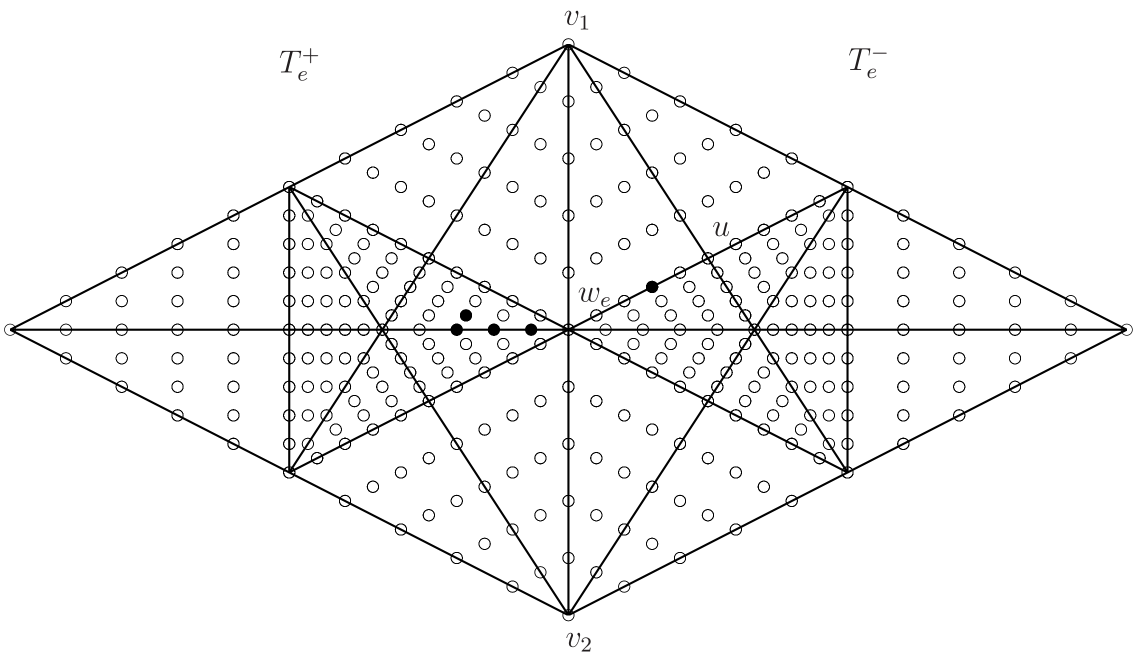

Figure 3: Domain points corresponding to $\tilde{M}_e$ are marked with filled circles.

Theorem 2. *The dimension of S_n is given by*

$$\dim S_n = 10\#\mathcal{V}_n + \#\tilde{\mathcal{V}}_n + 4\#\mathcal{E}_n + \#\tilde{\mathcal{E}}_n. \tag{4}$$

Moreover, the set

$$M_n = \bigcup_{v\in\mathcal{V}_n\setminus\tilde{\mathcal{V}}_n} M_v \cup \bigcup_{v\in\tilde{\mathcal{V}}_n} \tilde{M}_v \cup \bigcup_{e\in\mathcal{E}_n\setminus\tilde{\mathcal{E}}_n} M_e \cup \bigcup_{e\in\tilde{\mathcal{E}}_n} \tilde{M}_e$$

is a stable local minimal determining set for S_n.

The proof of this theorem will be given in Section 4.

By restricting to a single Powell-Sabin-12 split T_{PS12}, we consider the space $S(T_{PS12})$ defined by

$$\begin{aligned} S(T_{PS12}) = \big\{ s \in S_5^2(T_{PS12}) :\ & \\ & s \in C^3(v_i) \text{ for all } i = 1,2,3, \\ & s \in C^3(w_i) \text{ for all } i = 1,2,3, \\ & s \text{ is } C^3 \text{ across the segment } \langle v_i, u_i\rangle, \langle u_i, v_T\rangle, \langle w_i, v_T\rangle, \text{ for all } i = 1,2,3 \big\}, \end{aligned}$$

where v_i, u_i, w_i are as in Figure 1. Clearly, $S(T_{PS12}) = S_0$ if Δ_0 consists of just one triangle.

Let $T_1 = \langle v_1, w_3, u_1\rangle$, $T_2 = \langle v_2, w_1, u_2\rangle$, $T_3 = \langle v_3, w_2, u_3\rangle$, $T_4 = \langle v_T, w_2, u_1\rangle$, $T_5 = \langle v_T, w_3, u_2\rangle$, $T_6 = \langle v_T, w_1, u_3\rangle$, and let

$$M_v = \bigcup_{i=1}^{3} (D_3(v_i) \cap T_i), \qquad M_e = \bigcup_{i=4}^{6} \big\{ \xi^{T_i}_{1,4,0}, \xi^{T_i}_{2,3,0}, \xi^{T_i}_{3,2,0}, \xi^{T_i}_{2,2,1} \big\}.$$

Theorem 2 specialised to the case of $S(T_{PS12})$ gives the following corollary, see Figure 4 for an illustration.

Corollary 3. *The dimension of* $S(T_{PS12})$ *is 42. Moreover, the set* $M = M_v \cup M_e$ *is a stable minimal determining set for* $S(T_{PS12})$.

Remark 1. The spaces $S(T_{PS12})$ can be used to define non-nested C^2 macro-element spaces which in fact coincide with the spaces of C^2 Powell-Sabin-12 macro-elements constructed in [11] when the v_T is the barycentre of T. Note that our definition of $S(T_{PS12})$ is simpler than the corresponding space $S_2(T_{PS12})$ in [11].

4 Proof of Theorem 2

We start by providing two auxiliary results.

Let T_{PS6} be the Powell-Sabin-6 split of the triangle $T = \langle w_1, w_2, w_3\rangle$ which lies inside the Powell-Sabin-12 split in Figure 1 and is shown separately in Figure 5 for convenience. Recall that u_i is the midpoint of the edge opposite to w_i for $i = 1,2,3$, and $v_T = (w_1 + w_2 + w_3)/3$ is the barycentre of T. We consider the space $S_5^3(T_{PS6})$ of all C^3 piecewise quintics on T_{PS6}.

Let $T_1 = \langle v_T, w_1, u_3\rangle, T_2 = \langle v_T, w_2, u_1\rangle$ and $T_3 = \langle v_T, w_3, u_2\rangle$, and let

$$M = \bigcup_{i=1}^{3} (D_3(w_i) \cap T_i),$$

see Figure 6, where the points in M are marked with filled circles.

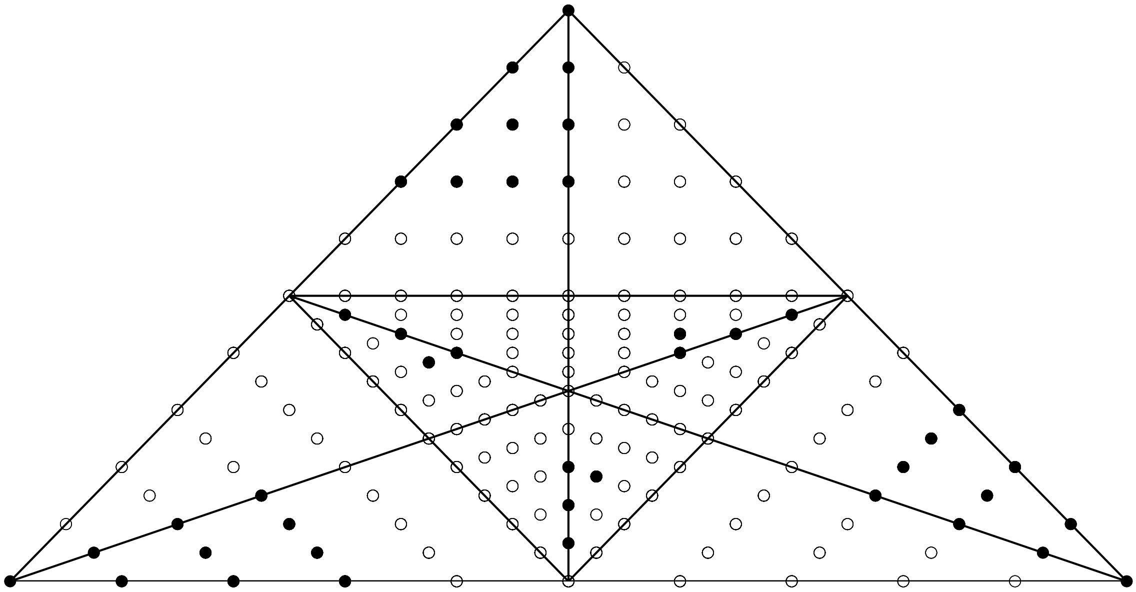

Figure 4: Minimal determining set for $S(T_{PS12})$ is indicated by filled circles.

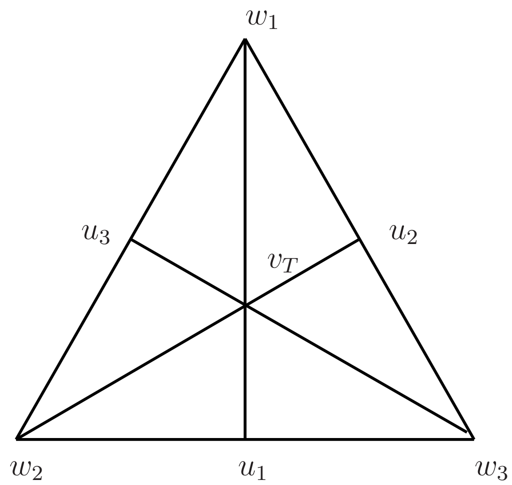

Figure 5: Powell-Sabin-6 Split

Lemma 4. *The dimension of* $S_5^3(T_{PS6})$ *is 30. Moreover, the above set* M *is a stable minimal determining set for* $S_5^3(T_{PS6})$.

Proof. The dimension of $S_5^3(T_{PS6})$ is easily obtained by [8, Theorem 9.3]. Let us show that M is a minimal determining set for $S_5^3(T_{PS6})$. For each $i = 1, 2, 3$, we use the C^3 smoothness at w_i to uniquely and stably compute the coefficients corresponding to all domain points in $D_3(w_i) \setminus M$ by [8, Lemma 5.10].

Next, for each edge $e_1 = \langle w_2, w_3 \rangle$, $e_2 = \langle w_1, w_3 \rangle$, $e_3 = \langle w_1, w_2 \rangle$ of T, we use the C^3 smoothness across the edge $\langle v_T, u_1 \rangle$, $\langle v_T, u_2 \rangle$, $\langle v_T, u_3 \rangle$, respectively, to determine the

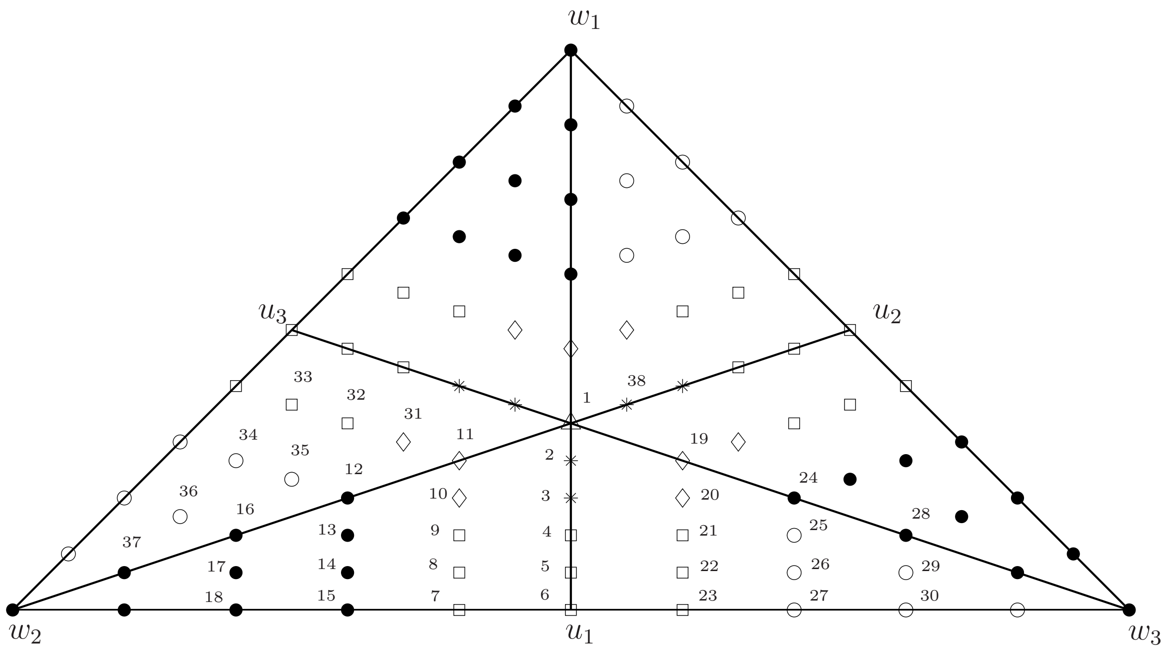

Figure 6: Minimal determining set M for $S_5^3(T_{PS6})$ is marked by filled circles.

coefficients corresponding to domain points in the set

$$E_2(e_i) := \{\xi : \operatorname{dist}(\xi, e_i) \le 2,\ \xi \notin D_3(w_i) \cup D_3(w_{i+1})\}, \quad i = 1, 2, 3.$$

(These coefficients are indicated by squares in Figure 6.) The C^3 smoothness across the edge $\langle v_T, u_i \rangle$ gives three smoothness conditions involving these coefficients which uniquely determine them as solutions of the corresponding linear system. For example, the barycentric coordinates of w_3 relative to $\langle w_2, u_1, v_T \rangle$ are given by $(-1, 2, 0)$ since u_1 is the midpoint of the edge $\langle w_2, w_3 \rangle$. Hence, the three smoothness conditions across the edge $\langle v_T, u_1 \rangle$ involving the coefficients on the edge $\langle w_2, w_3 \rangle$ are given by

$$\begin{aligned}
C_{23} &= -C_7 + 2C_6, \\
C_{27} &= C_{15} - 4C_7 + 4C_6, \\
C_{30} &= -C_{18} + 6C_{15} - 12C_7 + 8C_6,
\end{aligned}$$

where the coefficients C_i of a spline $s \in S_5^3(T_{PS6})$ are numbered as in Figure 6. By solving this linear system of equations with respect to C_7, C_{23} and C_6, we get

$$\begin{aligned}
C_7 &= -\tfrac{1}{4}C_{18} - \tfrac{1}{4}C_{30} + C_{15} + \tfrac{1}{2}C_{27}, \\
C_{23} &= -\tfrac{1}{4}C_{18} - \tfrac{1}{4}C_{30} + \tfrac{1}{2}C_{15} + C_{27}, \\
C_6 &= -\tfrac{1}{4}C_{18} - \tfrac{1}{4}C_{30} + \tfrac{3}{4}C_{15} + \tfrac{3}{4}C_{27}.
\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}
C_8 &= -\tfrac{1}{4}C_{17} - \tfrac{1}{4}C_{29} + C_{14} + \tfrac{1}{2}C_{26}, \quad & C_{22} &= -\tfrac{1}{4}C_{17} - \tfrac{1}{4}C_{29} + \tfrac{1}{2}C_{14} + C_{26}, \\
C_5 &= -\tfrac{1}{4}C_{17} - \tfrac{1}{4}C_{29} + \tfrac{3}{4}C_{14} + \tfrac{3}{4}C_{26}, \quad & C_9 &= -\tfrac{1}{4}C_{16} - \tfrac{1}{4}C_{28} + C_{13} + \tfrac{1}{2}C_{25}, \\
C_{21} &= -\tfrac{1}{4}C_{16} - \tfrac{1}{4}C_{28} + \tfrac{1}{2}C_{13} + C_{25}, \quad & C_4 &= -\tfrac{1}{4}C_{16} - \tfrac{1}{4}C_{28} + \tfrac{3}{4}C_{13} + \tfrac{3}{4}C_{25}.
\end{aligned}$$

The other coefficients indicated by squares in Figure 6 can be found in the same way.

By taking into account the C^3 smoothness condition across the edges $\langle w_1, v_T\rangle$, $\langle w_2, v_T\rangle$, $\langle w_3, v_T\rangle$, we compute the yet unknown coefficients corresponding to the domain points on the rings $R_4(w_1)$, $R_4(w_2)$ and $R_4(w_3)$, respectively, compare [8, Lemma 2.30]. These coefficients are indicated by diamonds in Figure 6. For example, the barycentric coordinates of u_1 relative to $\langle u_3, w_2, v_T\rangle$ are $(-1, \frac{1}{2}, \frac{3}{2})$, and hence the three smoothness conditions across the edge $\langle w_2, v_T\rangle$ involving the coefficients in the ring $R_4(w_2)$ are given by

$$C_{10} = -C_{31} + \tfrac{1}{2}C_{12} + \tfrac{3}{2}C_{11}, \quad C_9 = C_{32} - C_{35} - 3C_{31} + \tfrac{1}{4}C_{16} + \tfrac{3}{2}C_{12} + \tfrac{9}{4}C_{11},$$
$$C_8 = -C_{33} + \tfrac{3}{2}C_{34} + \tfrac{9}{2}C_{32} - \tfrac{3}{4}C_{36} - \tfrac{9}{2}C_{35} - \tfrac{27}{4}C_{31} + \tfrac{1}{8}C_{37} + \tfrac{9}{8}C_{16} + \tfrac{27}{8}C_{12} + \tfrac{27}{8}C_{11}.$$

By solving the linear system involving the above equations, we get

$$C_{10} = \tfrac{4}{3}C_9 + \tfrac{2}{3}C_{32} - \tfrac{2}{3}C_{35} + \tfrac{1}{6}C_{16} - \tfrac{4}{9}C_8 - \tfrac{4}{9}C_{33} + \tfrac{2}{3}C_{34} - \tfrac{1}{3}C_{12} + \tfrac{1}{18}C_{37} - \tfrac{1}{3}C_{36},$$
$$C_{11} = \tfrac{4}{3}C_9 + \tfrac{4}{3}C_{32} + \tfrac{1}{3}C_{16} - \tfrac{16}{27}C_8 - \tfrac{16}{27}C_{33} + \tfrac{8}{9}C_{34} - \tfrac{4}{9}C_{36} + \tfrac{2}{27}C_{37} - \tfrac{4}{3}C_{35},$$
$$C_{31} = \tfrac{2}{3}C_9 + \tfrac{4}{3}C_{32} - \tfrac{4}{3}C_{35} + \tfrac{1}{2}C_{12} - \tfrac{4}{9}C_8 - \tfrac{4}{9}C_{33} + \tfrac{2}{3}C_{34} - \tfrac{1}{3}C_{36} + \tfrac{1}{18}C_{37} + \tfrac{1}{3}C_{16}.$$

By using C^1 smoothness across the edges $\langle u_1, v_T\rangle$, $\langle u_2, v_T\rangle$, $\langle u_3, v_T\rangle$ we compute the remaining undetermined coefficients corresponding to the domain points at distances three and four from $\langle w_1, w_2\rangle$, $\langle w_2, w_3\rangle$, $\langle w_3, w_1\rangle$. These coefficients are marked by stars in Figure 6. For instance, since that the coefficients C_{19} and C_{20} are already known, we compute C_2 and C_3 using the formulas

$$C_2 = \tfrac{1}{2}C_{11} + \tfrac{1}{2}C_{19}, \quad C_3 = \tfrac{1}{2}C_{10} + \tfrac{1}{2}C_{20}.$$

Finally, the only remaining undetermined coefficient at v_T, marked by a triangle in Figure 6, can be computed by using for example the univariate C^1 smoothness condition along the line $\langle u_2, w_2\rangle$, which gives

$$C_1 = \tfrac{2}{3}C_{11} + \tfrac{1}{3}C_{38}.$$

We have shown that M is a determining set for $S_5^3(T_{PS6})$. The set M is minimal since its cardinality is equal to the dimension of $S_5^3(T_{PS6})$. The stability of M is obvious in view of [8, Lemma 5.10] and the above explicit formulas. □

Remark 2. The space $S_5^3(T_{PS6})$ coincides with the space $S_2(T_{PS6})$ of [8, Theorem 7.9], where the vertex v_T is placed at the barycentre of T.

Lemma 5. *Let Δ be the triangulation shown in Figure 7 with six vertices $v_1, \ldots, v_6$,, where $v_4 = (v_3 + v_5)/2$, $v_2 = (3v_1 + v_3)/4$ and $v_6 = (3v_1 + v_5)/4$. Let $T = \langle v_1, v_4, v_6\rangle$. Let $M = D_1(v_3) \cup D_1(v_5) \cup M_e \subset D_{3,\Delta}$, where*

$$M_e = \{\xi^T_{1,2,0}, \xi^T_{2,1,0}, \xi^T_{3,0,0}, \xi^T_{2,0,1}\}.$$

Then M is a stable minimal determining set for the space $\mathbb{P}_3$ of cubic polynomials regarded as a subspace of $S_3^0(\Delta)$.

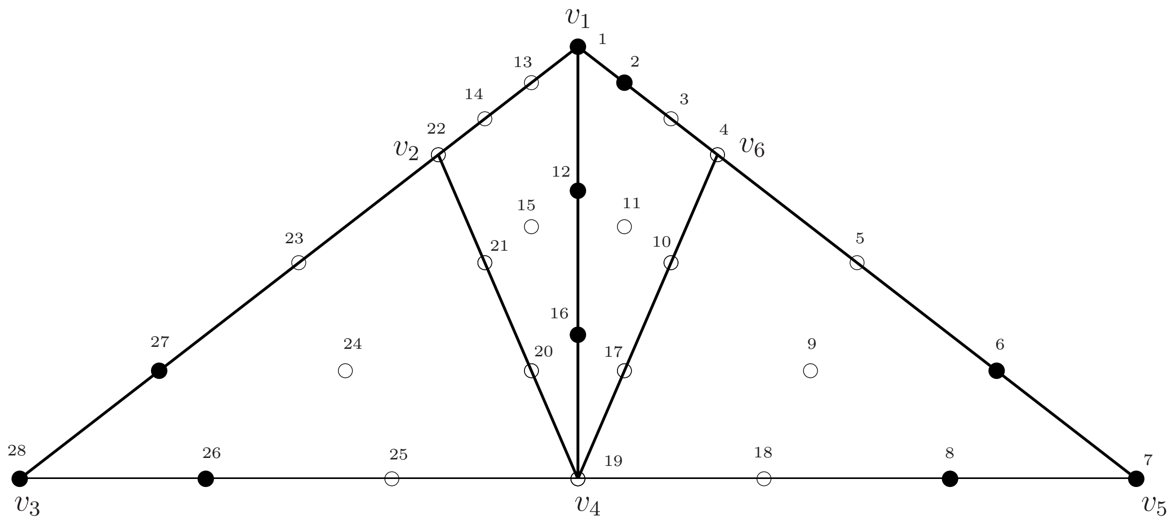

Figure 7: Triangulation of Lemma 5, where the minimal determining set M is indicated by filled circles.

Proof. The lemma follows from a more general statement given in [11, Lemma 4.1]. We provide a (somewhat different) proof in order to work out explicit formulas for the B-coefficients. We see that $\#M = \dim \mathbb{P}_3 = 10$. Hence we only need to show that if we set the coefficients of $s \in \mathbb{P}_3$ corresponding to $\xi \in M$, then all other coefficients are stably determined. Suppose the coefficients of s are numbered as in Figure 7. Then using the C^1 smoothness across the common edge $\langle v_1, v_4 \rangle$ of two triangle $T_1 = \langle v_2, v_4, v_1 \rangle$ and $T_2 = \langle v_6, v_1, v_4 \rangle$, where the barycentric coordinates of v_6 relative to T_1 are given by $(-1, \frac{1}{2}, \frac{3}{2})$, we obtain

$$C_{13} = \tfrac{1}{2} C_{12} + \tfrac{3}{2} C_1 - C_2.$$

By using the univariate C^1, C^2 and C^3 smoothness conditions along the edge $\langle v_1, v_3 \rangle$, where the barycentric coordinates of v_3 relative to $\langle v_1, v_2 \rangle$ are given by $(-3, 4)$ and solving the linear system involving the three equations, we obtain

$$\begin{aligned} C_{14} &= -\tfrac{9}{16} C_1 - \tfrac{1}{48} C_{28} + \tfrac{3}{2} C_{13} + \tfrac{1}{12} C_{27}, \\ C_{23} &= -\tfrac{27}{16} C_1 - \tfrac{1}{16} C_{28} + \tfrac{9}{4} C_{13} + \tfrac{1}{2} C_{27}, \\ C_{22} &= -\tfrac{27}{32} C_1 - \tfrac{1}{32} C_{28} + \tfrac{27}{16} C_{13} + \tfrac{3}{16} C_{27}. \end{aligned}$$

By symmetry, similar formulas hold for C_3, C_4 and C_5, and by using the univariate C^1, C^2 and C^3 smoothness conditions along the edge $\langle v_3, v_5 \rangle$ and solving the corresponding linear system, we get the formulas

$$\begin{aligned} C_{25} &= -\tfrac{1}{4} C_{28} - \tfrac{1}{4} C_7 + C_{26} + \tfrac{1}{2} C_8, \\ C_{18} &= -\tfrac{1}{4} C_{28} - \tfrac{1}{4} C_7 + \tfrac{1}{2} C_{26} + C_8, \\ C_{19} &= -\tfrac{1}{4} C_{28} - \tfrac{1}{4} C_7 + \tfrac{3}{4} C_{26} + \tfrac{3}{4} C_8. \end{aligned}$$

In the next step we compute C_{20} and C_{17} by C^1 smoothness conditions across edges $\langle v_2, v_4\rangle$ and $\langle v_4, v_6\rangle$, respectively,

$$C_{20} = \tfrac{1}{4}C_{25} + \tfrac{3}{4}C_{16}, \quad C_{17} = \tfrac{1}{4}C_{18} + \tfrac{3}{4}C_{16}.$$

By the C^1 and C^2 smoothness conditions across the edge $\langle v_1, v_4\rangle$, as well as the C^2 smoothness conditions across the edges $\langle v_2, v_4\rangle$ and $\langle v_4, v_6\rangle$, we obtain the system of equations

$$\begin{aligned}
C_{11} &= -C_{15} + \tfrac{1}{2}C_{16} + \tfrac{3}{2}C_{12},\\
C_{10} &= C_{21} - C_{20} - 3C_{15} + \tfrac{1}{4}C_{19} + \tfrac{3}{2}C_{16} + \tfrac{9}{4}C_{12},\\
C_{26} &= 9C_{12} - 24C_{15} + 16C_{21},\\
C_8 &= 16C_{12} - 24C_{11} + 9C_{10},
\end{aligned}$$

which can be solved with respect to C_{10}, C_{11}, C_{15} and C_{21} to give

$$\begin{aligned}
C_{10} &= -\tfrac{1}{7}C_8 + C_{12} + \tfrac{12}{7}C_{16} + \tfrac{4}{7}C_{19} - \tfrac{16}{7}C_{20} + \tfrac{1}{7}C_{26},\\
C_{11} &= -\tfrac{2}{21}C_8 + \tfrac{25}{24}C_{12} + \tfrac{9}{14}C_{16} + \tfrac{3}{14}C_{19} - \tfrac{6}{7}C_{20} + \tfrac{3}{56}C_{26},\\
C_{15} &= \tfrac{2}{21}C_8 + \tfrac{11}{24}C_{12} - \tfrac{1}{7}C_{16} - \tfrac{3}{14}C_{19} + \tfrac{6}{7}C_{20} - \tfrac{3}{56}C_{26},\\
C_{21} &= \tfrac{1}{7}C_8 + \tfrac{1}{8}C_{12} - \tfrac{3}{14}C_{16} - \tfrac{9}{28}C_{19} + \tfrac{9}{7}C_{20} - \tfrac{1}{56}C_{26}.
\end{aligned}$$

Finally, using C^1 smoothness across the edges $\langle v_4, v_6\rangle$ and $\langle v_2, v_4\rangle$, we get

$$C_9 = -3C_{11} + 4C_{10}, \quad C_{24} = -3C_{15} + 4C_{21},$$

which completes the proof that M is a determining set for $\mathbb{P}_3$. The stability is again obvious in view of the explicit formulas used. □

Proof of Theorem 2. To see that M_n is a stable minimal determining set, we show that we can set the coefficients $\{c_\xi\}_{\xi\in M_n}$ of a spline $s \in S_n$ to arbitrary values, and that all other coefficients of s are then uniquely and stably determined.

First, we show how the coefficients in $\mathcal{D}_{5,\Delta_n^*} \setminus M_n$ can be computed. For each $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, using C^3 smoothness conditions at v and the coefficients corresponding to M_v, we can uniquely compute the coefficients of s corresponding to all domain points in $D_3(v)$ by [8, Lemma 5.10]. For each $v \in \tilde{\mathcal{V}}_n$, using the C^3 smoothness of $s|_{T_v^+}$ at v and the coefficients in $M_v \cap T_v^+$, we compute the coefficients of s corresponding to all domain points in $D_3(v) \cap T_v^+$. By using C^2 smoothness across the common edge e_v of T_v^+ and T_v^-, we can compute the coefficients corresponding to $D_2(v) \cap T_v^-$. Then by C^3 smoothness at v inside T_v^- and the coefficient corresponding to the domain point $\xi_{2,3,0}^{\tilde{T}_v}$ inside T_v^- we can compute the remaining coefficients corresponding to all domain points in $R_3(v) \cap T_v^-$.

For each $e = \langle u, v\rangle \in \mathcal{E}_n \setminus \tilde{\mathcal{E}}_n$, using the coefficients corresponding to M_e, we now apply Lemma 5 to determine the coefficients of s corresponding to domain points in the disk $D_3(w_e)$, where w_e is the midpoint of e. Due to the C^3 smoothness at w_e, we can regard the coefficients of s in the disk as coefficients of a polynomial g of degree 3. Lemma 5 ensures that we can set the coefficients of s corresponding to domain points in

M_e to arbitrary values, and that all coefficients corresponding to the remaining domain points in $D_3(w_e)$ are uniquely and stably determined.

For each $e = \langle u, v \rangle \in \tilde{\mathcal{E}}_n$, using the C^3 smoothness of $s|_{T_e^+}$ at the midpoint w_e of e and the coefficients corresponding to $\{\xi_{3,2,0}^{\hat{T}_e^3}, \xi_{2,3,0}^{\hat{T}_e^3}, \xi_{2,2,1}^{\hat{T}_e^3}, \xi_{1,4,0}^{\hat{T}_e^3}\}$ inside T_e^+ we can compute the coefficients of s corresponding to domain points in the disk $D_3(w_e) \cap T_e^+$ by Lemma 5 as described previously. Using the C^2 smoothness across the common edge e of T_e^+ and T_e^- we can compute the coefficients corresponding to $D_2(w_e) \cap T_e^-$. Then using the C^3 smoothness condition supported inside T_e^- and the coefficient corresponding to the point $\xi_{2,3,0}^{\tilde{T}_e}$ we can compute the remaining coefficients corresponding to domain points in $R_3(w_e) \cap T_e^-$. For each type-1 edge $e = \langle u, v \rangle$, by taking account of the C^3 smoothness across the edge $e = \langle u, v \rangle$ we can now compute the three central coefficients in the ring $R_4(v)$. Note that in practice these coefficients can be more conveniently computed by using C^1 smoothness conditions across the edges of the form $\langle w_i, w_{i+1} \rangle$, see Remark 3.

We now show that the coefficients corresponding to the remaining domain points are uniquely determined. These remaining domain points lie inside triangles of the form $T = \langle w_1, w_2, w_3 \rangle$, where $w_i \in \mathcal{W}_n$. Let T_{PS6} be the Powell-Sabin-6 split of T, see Figure 5. We have already determined all coefficients corresponding to domain points in the disks $D_3(w_i)$ for $i = 1, 2, 3$. Now we can apply Lemma 4 to uniquely and stably determine all coefficients of s corresponding to the remaining domain points in T.

We have thus shown that M is a determining set for S_n. To complete the proof, we need to show that the six central C^1, C^2 smoothness conditions across the edges $\langle w_i, w_{i+1} \rangle$, $i = 1, 2, 3$ are satisfied. Indeed, all other smoothness conditions are either used in the above computation or are satisfied in view of Lemmas 4 and 5.

To check these conditions we will only look at the section $\langle v_1, w_3, v_T, w_2 \rangle$ of the triangle $T = \langle v_1, v_2, v_3 \rangle$ as shown in Figure 8, where we indicate the domain points of the 5×7 grid around u_1 by double integer indices (i, j) with the origin $(0, 0)$ at u_1 and the row of indices with $j = 0$ on the edge $\langle w_2, w_3 \rangle$. The coefficient corresponding to (i, j) is denoted by $C_{i,j}$. All smoothness conditions that need verification are supported within this grid. The smoothness conditions on the other two sections of the triangle can be checked in the same way.

Let (γ, β, δ) be the barycentric coordinates of v_T relative to $\langle v_1, w_3, u_1 \rangle$. Thus $\beta = 0$ and $v_T = \gamma v_1 + \delta u_1$ where $\gamma = -\frac{1}{3}$ and $\delta = \frac{4}{3}$. We first write down the known C^1, C^2 and C^3 smoothness conditions in rows $j = -2, -1, 0, 1, 2$ of the grid,

$$\begin{aligned} C_{3,j} &= 8C_{0,j} - 12C_{-1,j} + 6C_{-2,j} - C_{-3,j}, \\ C_{2,j} &= 4C_{0,j} - 4C_{-1,j} + C_{-2,j}, \\ C_{1,j} &= 2C_{0,j} - C_{-1,j}. \end{aligned}$$

By solving this linear system for $C_{-1,j}, C_{0,j}, C_{1,j}$, we obtain for $j = -2, -1, 0, 1, 2$

$$C_{-1,j} = \tfrac{1}{4}(2C_{2,j} - C_{3,j} + 4C_{-2,j} - C_{-3,j}), \tag{5}$$

$$C_{0,j} = \tfrac{1}{4}(3C_{2,j} - C_{3,j} + 3C_{-2,j} - C_{-3,j}), \tag{6}$$

$$C_{1,j} = C_{2,j} - \tfrac{1}{4}C_{3,j} + \tfrac{1}{2}C_{-2,j} - \tfrac{1}{4}C_{-3,j}. \tag{7}$$

We write down the four known C^1 smoothness conditions across row 0 of the grid

$$C_{i,-1} = \gamma C_{i,1} + \delta C_{i,0}, \quad i = -3, -2, 2, 3. \tag{8}$$

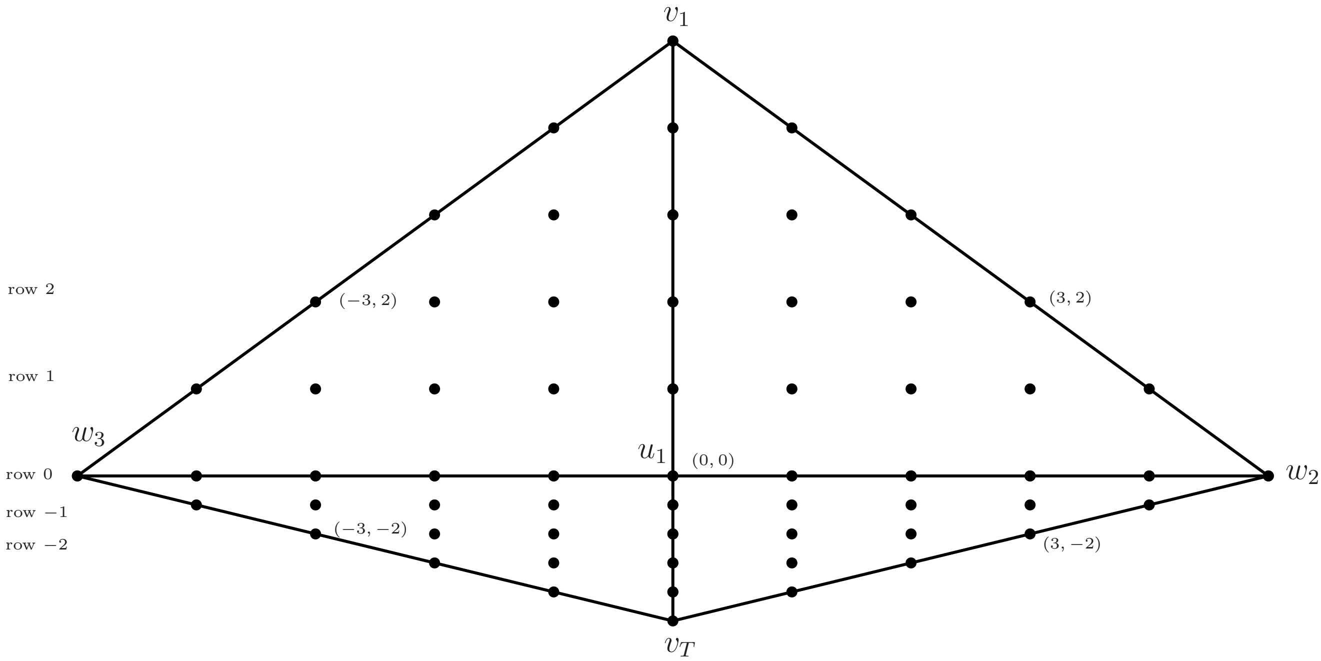

Figure 8: The 5×7 grid around u_1.

By replacing $C_{-3,-1}, C_{-2,-1}, C_{2,-1}, C_{3,-1}$ in (5)–(7) with expressions in (8) and collecting the terms with coefficients γ and δ, we obtain

$$C_{i,-1} = \gamma C_{i,1} + \delta C_{i,0}, \quad i = -1, 0, 1,$$

which confirms the three remaining C^1 smoothness conditions across row 0. Similarly, by replacing $C_{3,2}, C_{2,2}, C_{-2,2}, C_{-3,2}$ in (5)–(7) with expressions in the four known C^2 smoothness conditions across row 0,

$$C_{i,2} = \gamma^2 C_{i,0} + 2\gamma\delta C_{i,-1} + \delta^2 C_{i,-2}, \quad i = -3, -2, 2, 3,$$

we verify the three remaining C^2 smoothness conditions across row 0,

$$C_{i,2} = \gamma^2 C_{i,0} + 2\gamma\delta C_{i,-1} + \delta^2 C_{i,-2}, \quad i = -1, 0, 1.$$

We have shown that M_n is a stable local minimal determining set for S_n. Hence, the dimension of S_n is equal to the cardinality of M_n, which is easily seen to be the number in (4). □

Remark 3. Note that in practice the three central coefficients in the ring $R_4(v_1)$ are more conveniently computed by using C^1 smoothness conditions across the edge $\langle w_3, w_2 \rangle$ rather than by C^1, C^2 and C^3 smoothness conditions across $\langle v_1, u_1 \rangle$ according to the above proof. Indeed, in the notation of Figure 8 these coefficients are given by

$$C_{-1,1} = 4C_{-1,0} - 3C_{-1,-1}, \quad C_{0,1} = 4C_{0,0} - 3C_{0,-1}, \quad C_{1,1} = 4C_{1,0} - 3C_{1,-1}.$$

5 A nodal minimal determining set for S_n

As usual for macro-element spaces, we provide a stable nodal minimal determining set for S_n and an error bound for the corresponding Hermite interpolation operator.

Recall that a linear functional λ is called a *nodal functional* provided that λf is a combination of values and/or derivatives of f at some point η. A collection $\mathcal{N} = \{\lambda\}_{i=1}^N$ is called a *nodal determining set* for a spline space S if $\lambda s = 0$ for all $\lambda \in \mathcal{N}$ implies $s \equiv 0$. Moreover, $\mathcal{N}$ is called a *nodal minimal determining set* (NMDS) for S if there is no smaller nodal determining set. We refer to [8, Section 5.9] for further details on nodal determining sets.

Let (u_x, u_y) and (v_x, v_y) be the Cartesian coordinates of u and v, respectively. Then the directional derivative of s at $(x, y) \in T$ with respect to the (directed) edge e is given by

$$D_e s(x, y) = (v_x - u_x) D_x s(x, y) + (v_y - u_y) D_y s(x, y).$$

Let $e^\perp$ be the directed segment obtained rotating e ninety degrees in the counterclockwise direction. We write $D_{e^\perp} s$ for the directional derivative of s associated with $e^\perp$. The linear functional evaluating at $\xi \in \Omega$ any function f continuous at ξ will be denoted by δ_ξ.

Lemma 6. *Let Δ be the triangulation shown in Figure 7, where $v_4 = (v_3 + v_5)/2$, $v_2 = (3v_1 + v_3)/4$ and $v_6 = (3v_1 + v_5)/4$. The set*

$$N = N_{v_3} \cup N_{v_4} \cup N_{v_5}$$

is a nodal determining set for $\mathbb{P}_3$, where

1) $N_{v_3} = \{\delta_{v_3}, \delta_{v_3} D_x, \delta_{v_3} D_y\}$,

2) $N_{v_5} = \{\delta_{v_5}, \delta_{v_5} D_x, \delta_{v_5} D_y\}$,

3) $N_{v_4} = \{\delta_{v_4} D_{e_1}, \delta_{v_4} D_{e_1}^2, \delta_{v_4} D_{e_2} D_{e_1}^2, \delta_{v_4} D_{e_1}^3\}$,

with $e_1 := \langle v_4, v_1 \rangle$ and $e_2 := \langle v_4, v_5 \rangle$.

Proof. It is clear that the cardinality of N is equal to the dimension of $\mathbb{P}_3$. Thus to prove that N is a nodal minimal determining set, we just need to show that given the values of $\{\lambda s\}_{\lambda \in N}$ all B-coefficients of $s \in \mathbb{P}_3$ can be determined. Suppose the coefficients of $s \in \mathbb{P}_3$ are numbered as in Figure 7. Using the data $\{\delta_{v_3} s, \delta_{v_3} D_x s, \delta_{v_3} D_y s\}$ at v_3 we can compute the coefficients C_{26}, C_{27} and C_{28} by [8, Theorem 2.19]. Similarly, using the data $\{\delta_{v_5} s, \delta_{v_5} D_x s, \delta_{v_5} D_y s\}$ at v_5 we compute the coefficients C_6, C_7 and C_8. Using the data $\{\delta_{v_4} D_{e_1} s, \delta_{v_4} D_{e_1}^2 s, \delta_{v_4} D_{e_1}^3 s\}$ we can compute the coefficients C_{16}, C_{12} and C_1 by [8, Lemma 2.20], that is, using the formulas

$$\begin{aligned} \delta_{v_4} D_{e_1} s &= -3C_{19} + 3C_{16}, \\ \delta_{v_4} D_{e_1}^2 s &= 6C_{19} - 12C_{16} + 6C_{12}, \\ \delta_{v_4} D_{e_1}^3 s &= -6C_{19} + 18C_{16} - 18C_{12} + 6C_1, \end{aligned}$$

where C_{19} can be computed using the three univariate smoothness conditions along the edge $\langle v_3, v_5 \rangle$ as in the proof of Lemma 5.

Let $e = \langle v_4, v_6 \rangle$. Since $v_6 - v_4 = \frac{1}{4}(v_5 - v_4) + \frac{3}{4}(v_1 - v_4)$, according to [8, (2.36)], we can use the data $\delta_{v_4} D_{e_2} D_{e_1}^2 s$ to compute the coefficient C_2 from the relation

$$\frac{1}{4}\delta_{v_4} D_{e_2} D_{e_1}^2 s + \frac{3}{4}\delta_{v_4} D_{e_1}^3 s = \delta_{v_4} D_e D_{e_1}^2 s = -6C_{19} + 12C_{16} + 6C_{12} + 12C_{11} - 6C_{17} - 6C_2,$$

where the coefficients C_{11} and C_{17} are computed as in the proof of Lemma 5.

At this point we have determined all coefficients corresponding to domain points in the minimal determining set M of Lemma 5, and it follows from that lemma that all other coefficients are also determined. □

Theorem 7. *The set*

$$N_n = \bigcup_{v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n} N_v \cup \bigcup_{v \in \tilde{\mathcal{V}}_n} \tilde{N}_v \cup \bigcup_{e \in \mathcal{E}_n \setminus \tilde{\mathcal{E}}_n} N_e \cup \bigcup_{e \in \tilde{\mathcal{E}}_n} \tilde{N}_e$$

is a nodal minimal determining set for S_n, where

1) $N_v = \{\delta_v D_x^\alpha D_y^\beta,\ 0 \le \alpha + \beta \le 3\}$,

2) $\tilde{N}_v = \{\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta,\ 0 \le \alpha + \beta \le 3,\ \beta \le 2\} \cup \{\delta_v^+ D_{e_v^\perp}^3, \delta_v^- D_{e_v^\perp}^3\}$,

3) $N_e = \left\{\delta_{w_e} D_{e^\perp}, \delta_{w_e} D_{e^\perp}^2, \delta_{w_e} D_e D_{e^\perp}^2, \delta_{w_e} D_{e^\perp}^3\right\}$,

4) $\tilde{N}_e = \left\{\delta_{w_e} D_{e^\perp}, \delta_{w_e} D_{e^\perp}^2, \delta_{w_e} D_e D_{e^\perp}^2, \delta_{w_e}^+ D_{e^\perp}^3, \delta_{w_e}^- D_{e^\perp}^3\right\}$,

where $\delta_v^\pm f := \delta_v(f|_{T_v^\pm})$ and $\delta_{w_e}^\pm f := \delta_{w_e}(f|_{T_e^\pm})$, and w_e denotes the midpoint of the edge e.

Proof. It is clear that the cardinality of N_n is equal to the dimension of S_n as given in (4). Thus to prove that N_n is a nodal minimal determining set, we just need to show that given the values of $\{\lambda s\}_{\lambda \in N_n}$ all B-coefficients of $s \in S_n$ can be determined.

For every vertex $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, we can compute all coefficients corresponding to domain points in the disk $D_3(v)$ directly from the data in N_v by [8, Theorem 2.19].

For every vertex $v \in \tilde{\mathcal{V}}_n$, we can compute all the coefficients of s corresponding to domain points in the disk $D_3(v)$ from the data in $\tilde{N}_v$. That is, using the data $\{\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta s,\ 0 \le \alpha + \beta \le 3,\ \beta \le 2\} \cup \{\delta_v^+ D_{e_v^\perp}^3 s\}$, we can compute the coefficients corresponding to domain points in $D_3(v) \cap T_v^+$ by [8, Theorem 2.19]. Then using the coefficients corresponding to domain points in $D_3(v) \cap T_v^+$ and C^2 smoothness conditions across the edge e_v, we can compute the coefficients corresponding to domain points in $D_2(v) \cap T_v^-$. Now using the data $\{\delta_v^- D_{e_v^\perp}^3 s\}$ and C^3 smoothness conditions inside T_v^-, we can compute all the remaining coefficients corresponding to domain points in $R_3(v) \cap T_v^-$.

Given an edge $e = \langle v', v'' \rangle$ in $\mathcal{E}_n \setminus \tilde{\mathcal{E}}_n$, let w_e be its midpoint. We now compute all coefficients of s corresponding to domain points in $D_3(w_e)$. By the C^3 smoothness at w_e, as in the proof of Theorem 2, these coefficients can be regarded as the coefficients of a polynomial g of degree 3. Hence, it follows from Lemma 6 that all B-coefficients in $D_3(w_e)$ are determined by the known B-coefficients in the sets $D_3(w_e) \cap D_3(v')$ and $D_3(w_e) \cap D_3(v'')$ and the nodal data in N_e.

Given an edge e in $\tilde{\mathcal{E}}_n$, we can compute the coefficients of s corresponding to the domain points in the disk $D_3(w_e)$ from the data in $\tilde{N}_e$. That is, using the data $\{\delta_{w_e} D_{e^\perp} s, \delta_{w_e} D_e D^2_{e^\perp} s, \delta_{w_e} D^2_{e^\perp} s, \delta^+_{w_e} D^3_{e^\perp} s\}$, we can compute the B-coefficients corresponding to $D_3(w_e) \cap T_e^+$ using the same argument as above. Then using the coefficients corresponding to domain points in $D_3(w_e) \cap T_e^+$, and C^2 smoothness conditions across the edge e we can compute all the coefficients corresponding to domain points in $D_2(w_e) \cap T_e^-$. Finally, using $\{\delta^-_{w_e} D^3_{e^\perp} s\}$ and C^3 smoothness conditions inside T_e^- we can compute all the remaining coefficients corresponding to domain points in $R_3(w_e) \cap T_e^-$. At this point we have determined all coefficients corresponding to domain points in the minimal determining set M_n of Theorem 2, and it follows from that theorem that all other coefficients are also determined. □

Corollary 8. *The set*

$$N = N_v \cup N_e$$

is a nodal minimal determining set for $S(T_{PS12})$, where

1) $N_v = \bigcup_{i=1}^3 \{\delta_{v_i} D_x^\alpha D_y^\beta,\ 0 \le \alpha + \beta \le 3\}$,

2) $N_e = \bigcup_{i=1}^3 \{\delta_{w_{e_i}} D_{e_i^\perp}, \delta_{w_{e_i}} D_{e_i} D^2_{e_i^\perp}, \delta_{w_{e_i}} D^2_{e_i^\perp}, \delta_{w_{e_i}} D^3_{e_i^\perp}\}$,

v_i are the three vertices of T, $e_1 := \langle v_1, v_2 \rangle$, $e_2 := \langle v_2, v_3 \rangle$, $e_3 := \langle v_3, v_1 \rangle$ and w_{e_i} denotes the midpoint of e_i.

By Theorem 7 for any function $f \in C^3(\Omega)$ and any $n = 0, 1, \ldots$, there exists a unique spline $s_n(f) \in S_n$ that solves the Hermite interpolation problem

$$\lambda s = \lambda f, \quad \lambda \in N_n$$

The following error bound follows immediately by [8, Theorem 5.26] if we take into account that the uniform refinement used to generate the triangulation Δ_n halves the diameters of the triangles and that stability of the nodal minimal determining set N_n can be verified by usual arguments, see e.g. [11, p. 724].

Theorem 9. *For every $f \in C^r(\Omega)$, with $3 \le r \le 6$,*

$$|f - s_n(f)|_{W^k_\infty(\Omega)} \le \frac{K}{2^{n(r-k)}} |f|_{W^r_\infty(\Omega)},$$

for all $0 \le k < r$, where K depends only on the maximum diameter and the smallest angle of the triangles of the initial triangulation Δ_0, and $|\cdot|_{W^r_\infty(\Omega)}$ denotes the standard Sobolev seminorm on Ω.

Remark 4. It can be easily checked that the nodal determining sets of Theorem 7 are nested, that is, $N_n \subset N_{n+1}$, $n = 0, 1, 2, \ldots$. This fact may be useful for certain multilevel algorithms, see e.g. [1].

Remark 5. In developing the macroelement spaces of this paper, we have used P. Alfeld's software for examining determining set for the spline spaces, available from `http://www.math.utah.edu/~alfeld`.

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