C^2 piecewise cubic quasi-interpolants on a 6-direction mesh

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Abstract

We study two kinds of quasi-interpolants (abbr. QI) in the space of C^2 piecewise cubics in the plane, or in a rectangular domain, endowed with the highly symmetric triangulation generated by a uniform 6-direction mesh. It has been proved recently that this space is generated by the integer translates of two multi-box splines. One kind of QIs is of differential type and the other of discrete type. As those QIs are exact on the space of cubic polynomials, their approximation order is 4 for sufficiently smooth functions. In addition, they exhibit nice superconvergent properties at some specific points. Moreover, the infinite norms of the discrete QIs being small, they give excellent approximations of a smooth function and of its first order partial derivatives. The approximation properties of the QIs are illustrated by numerical examples.

1 Introduction

Spline quasi-interpolants (abbr. QI) are effective tools in bivariate approximation on uniform grids. Their main interest lies in the fact that they are obtained without solving any system of linear equations. Moreover, they have often an infinite norm of reasonable size and they can provide the best approximation order on spaces of smooth functions. Among the extensive literature on this topic, one can cite for example the treatises [2, 6, 7, 13, 22], the surveys [3, 4] and the articles [9, 19, 24, 25]. More specifically, in the domain of approximation of large sets of scattered data, they are often used in the second stage of the two-stage methods [27, 12, 10, 11]. In the present paper, we construct two kinds of quasi-interpolants in the space of C^2 piecewise cubics in the plane, or in a rectangular domain, endowed with the Powell-Sabin (PS) triangulation generated by a uniform 6-direction mesh. This space has been studied recently by different authors from different points of view, e.g. Chui & Jiang [8], Lian [23], Bettayeb [1]. It is generated by the integer translates of two *multi-box splines* φ_1 and φ_2 (Goodman [15]-[18]). The QIs of the first kind are of *differential type* (DQI) and have the form

$$Q^* f(x) := \sum_{\alpha\in\mathbb{Z}^2} [\mathcal{D}_1^* f(\alpha), \mathcal{D}_2^* f(\alpha)]\, \Phi(x-\alpha),$$

where $\Phi := [\varphi_1, \varphi_2]^T$ and the coefficients $\mathcal{D}_1^* f(\alpha)$ and $\mathcal{D}_2^* f(\alpha)$ are linear differential operators of the second or the fourth order. The QIs of the second kind are of *discrete type*

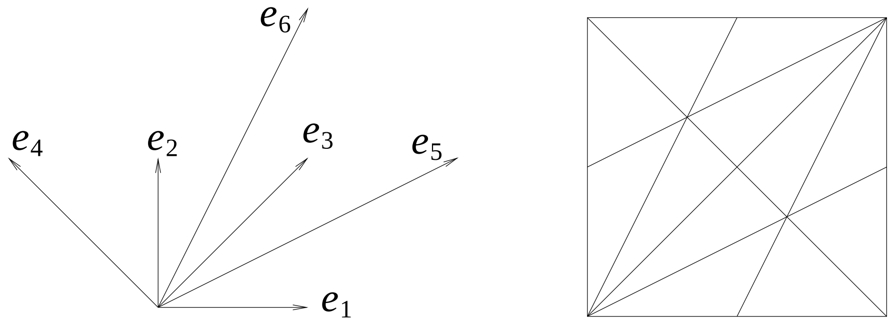

Figure 1: Left: Collection of vectors V. Right: Subdivision of a square cell.

(dQI) and have the form

$$Qf(x) := \sum_{\alpha\in\mathbb{Z}^2} [\mathcal{D}_1 f(\alpha), \mathcal{D}_2 f(\alpha)]\, \Phi(x-\alpha),$$

where the coefficients $\mathcal{D}_1 f(\alpha)$ and $\mathcal{D}_2 f(\alpha)$ are linear finite difference operators of the second (resp. the fourth) order which coincide with the previous differential operators on the spaces $\mathbb{P}_3$ (resp. $\mathbb{P}_5$) of bivariate cubic (resp. quintic) polynomials.

As those quasi-interpolants are exact on $\mathbb{P}_3$ (i.e. $Q^*p = Qp = p$ for all $p \in \mathbb{P}_3$), and φ_1, φ_2 are compactly supported, their approximation order is 4 for sufficiently smooth functions. In addition, they exhibit nice superconvergent properties at some specific points. Moreover, the infinite norms of the discrete quasi-interpolants being small, the latter gives excellent approximations of a smooth function and of its first order partial derivatives.

Here is an outline of the paper. In Section 2, we recall the main results on multi-box spline generators of the space of C^2 cubics on a 6-direction mesh. In Sections 3 and 4, thanks to the expansions of monomials in terms of translates of these multi-box splines given in [8], we define the differential and discrete quasi-interpolants in the whole plane. In Section 5, we give an upper bound of the norm of the dQI and error estimates for the approximation of a smooth function and of its partial derivatives. In Section 6, we initialize the study of discrete quasi-interpolants in a rectangular domain by giving examples of construction of coefficient functionals for some boundary multi-box splines. Finally, in Section 7, we give some numerical examples.

2 C^2 cubics on a 6-direction mesh

Define the vectors (see Figure 1 (left)):

$$e_1 := (1,0),\ e_2 := (0,1),\ e_3 := (1,1),\ e_4 := (-1,1),\ e_5 := (2,1),\ e_6 := (1,2).$$

With the collection of vectors $V := \{e_1, e_2, e_3, e_4, e_5, e_6\}$, one can associate the space $\mathcal{S}_3^2(V)$ of C^2 ppf of degree 3 on the triangulation generated by V (i.e. by all lines through points of $\mathbb{Z}^2$ in the directions of the vectors of V) which is often called the *6-direction mesh*. Any square cell is subdivided into two triangles by its main diagonal. Then each of these two triangles is subdivided into 6 subtriangles by its medians (Powell-Sabin split, see Figure 1 (right)).

The spaces $\mathcal{S}_3^2(V)$ have been recently studied in [8], where it is shown that $\mathcal{S}_3^2(V)$ can be generated by a pair of basis splines $\{\varphi_1, \varphi_2\}$, i.e. any element $S \in \mathcal{S}_3^2(V)$ can be expressed as

$$S = \sum_{\alpha\in\mathbb{Z}^2} \lambda_\alpha \varphi_1(\cdot - \alpha) + \sum_{\alpha\in\mathbb{Z}^2} \mu_\alpha \varphi_2(\cdot - \alpha).$$

As shown in [1], the generators φ_1, φ_2 are *multi-box splines* [18], and they can be defined as follows. The support of φ_1 is the unit hexagon H_1 with vertices $\{\pm e_1, \pm e_2, \pm e_3\}$ and φ_2 is defined as $\varphi_2 := \varphi_1(A^{-1})$, where $A := \begin{bmatrix} 2 & -1 \\ 1 & -2 \end{bmatrix}$. In other words,

$$\varphi_2(x_1, x_2) = \varphi_1\left(\frac{1}{3}(2x_1 - x_2), \frac{1}{3}(x_1 - 2x_2)\right).$$

Therefore supp(φ_2) is the hexagon with vertices $\{\pm e_4, \pm e_5, \pm e_6\}$. These functions can be defined by their Fourier transforms (see [1, Chapter 6]). Setting $\Phi := [\varphi_1, \varphi_2]^T$, $z_1 := e^{-iu_1}, z_2 := e^{-iu_2}$, this Fourier transform is equal to

$$\hat{\Phi}(u_1, u_2) := [\hat{\varphi}_1(u_1, u_2), \hat{\varphi}_2(u_1, u_2)]^T = [u_1, u_1{+}u_2]M(z_1, z_2)/u_1u_2(u_1^2{-}u_2^2)(u_1{+}2u_2)(2u_1{+}u_2),$$

where M is the matrix

$$M(z_1, z_2) := \begin{bmatrix} P_1(z_1, z_2) & P_2(z_1, z_2) \\ Q_1(z_1, z_2) & Q_2(z_1, z_2) \end{bmatrix},$$

whose entries are given by

$$\begin{aligned}
P_1(z_1, z_2) &:= z_1 + 2z_1z_2 + z_2 - z_1^{-1} - 2z_1^{-1}z_2^{-1} - z_2^{-1}, \\
P_2(z_1, z_2) &:= -3z_1^2z_2 + 3z_1^{-1}z_2^{-2} + 3z_1^{-2}z_2^{-1} - 3z_1z_2^2 \\
Q_1(z_1, z_2) &:= -2z_1 - z_1z_2 + z_2 + 2z_1^{-1} + z_1^{-1}z_2^{-1} - z_2^{-1} \\
Q_2(z_1, z_2) &:= 3z_1^2z_2 + 3z_1z_2^{-1} - 3z_1^{-2}z_2^{-1} - 3z_1^{-1}z_2.
\end{aligned}$$

Moreover, it is shown in [8, Section 2] that the space $\mathcal{S}_3^2(V)$ has full L_2-approximation order 4 for sufficiently smooth functions.

Quasi-interpolation operators

In this paper we are studying *quasi-interpolation operators* for $S_3^2(V)$ of the form

$$Qf = \sum_{\alpha\in\mathbb{Z}^2} [\lambda_\alpha f, \mu_\alpha f]\Phi(x - \alpha), \quad \Phi := [\varphi_1, \varphi_2]^T,$$

where $\lambda_\alpha, \mu_\alpha$ are linear functionals. A quasi-interpolation operator Q for $S_3^2(V)$ is said to possess *full approximation order* if

$$\|D^\gamma(Q_hf - f)\|_\infty \le Ch^{4-|\gamma|}\|D^4 f\|_\infty, \quad |\gamma| \le 3, \tag{1}$$

where C is a constant independent of h and f,

$$Q_hf(x) := Qf_h(x/h), \qquad f_h(x) = f(hx), \qquad h > 0,$$

$$D^\gamma := \partial_1^r \partial_2^s, \qquad \partial_1 := \frac{\partial}{\partial x_1}, \quad \partial_2 := \frac{\partial}{\partial x_2}, \qquad \gamma = (r,s) \in \mathbb{Z}_+^2, \qquad |\gamma| := r+s.$$

We are mostly interested in *discrete quasi-interpolants* (dQI), where the functionals $\lambda_\alpha f, \mu_\alpha f$ are linear combinations of a finite number of the values of f on the grid $\mathbb{Z}^2$. However, discrete quasi-interpolants are usually obtained by discretisation from suitable *differential quasi-interpolants* (DQI), whose functionals $\lambda_\alpha f, \mu_\alpha f$ include (higher order) derivatives of f at the grid points. In what follows we suggest two pairs of DQIs and associated with them dQIs: $Q^*, \overline{Q}^*$, and $Q^\#, \overline{Q}^\#$.

Using the values of multi-box splines φ_1, φ_2 at the grid $\mathbb{Z}^2$, we compute for future reference the following expressions for the values of Qf, $\partial_1 Qf$ and $\partial_2 Qf$ at the origin:

$$\begin{aligned}
Qf(0,0) &= \lambda_{0,0} f + \mu_{0,0} f + \frac{1}{9}(\mu_{1,0} f + \mu_{-1,0} f + \mu_{0,1} f + \mu_{0,-1} f + \mu_{1,1} f + \mu_{-1,-1} f),\\
\partial_1 Qf(0,0) &= \frac{1}{3}(\mu_{1,0} f - \mu_{-1,0} f) + \frac{1}{6}\left(\mu_{0,-1} f - \mu_{0,1} f + \mu_{1,1} f - \mu_{-1,-1} f\right),\\
\partial_2 Qf(0,0) &= \frac{1}{3}(\mu_{0,1} f - \mu_{0,-1} f) + \frac{1}{6}\left(\mu_{-1,0} f - \mu_{1,0} f + \mu_{1,1} f - \mu_{-1,-1} f\right).
\end{aligned}$$

3 Differential quasi-interpolants in $\mathbb{R}^2$

3.1 Differential quasi-interpolants of the second order

We normalize the multi-box splines so that their multi-integer translates sum to one, by taking

$$\bar\varphi_1 = \frac{1}{6}\varphi_1, \quad \bar\varphi_2 = \frac{1}{2}\varphi_2, \quad \bar\Phi := [\bar\varphi_1, \bar\varphi_2]^T.$$

According to [8], the expression of monomials $m_\gamma(x) := x_1^r x_2^s$, $\gamma = (r,s) \in \mathbb{Z}_+^2$, is given by the formula

$$m_\gamma(x) = \sum_\alpha \left(\sum_{\beta \le \gamma} \binom{\gamma}{\beta} \alpha^{\gamma-\beta} y_\beta \right) \bar\Phi(x-\alpha), \quad \gamma \in \mathbb{Z}_+^2, \quad |\gamma| \le 3, \tag{2}$$

where the vectors y_β are respectively defined by

$$y_{0,0} = [1,1], \quad y_{2,0} = y_{0,2} = \frac{1}{3}[1,-1], \quad y_{1,1} = \frac{1}{6}[1,-1],$$
$$y_\beta = [0,0] \quad \text{for} \quad |\beta| = 1 \text{ and } 3.$$

Therefore we obtain the expansions of monomials of $\mathbb{P}_3$ in terms of translates of multi-box splines:

$$1 = \sum_\alpha [1,1]\bar\Phi(x-\alpha), \quad x_1 = \sum_\alpha [\alpha_1, \alpha_1]\bar\Phi(x-\alpha), \quad x_2 = \sum_\alpha [\alpha_2, \alpha_2]\bar\Phi(x-\alpha),$$

$$x_1^2 = \sum_\alpha \left[\alpha_1^2 + \frac{1}{3}, \alpha_1^2 - \frac{1}{3}\right] \bar\Phi(x-\alpha), \quad x_1 x_2 = \sum_\alpha \left[\alpha_1\alpha_2 + \frac{1}{6}, \alpha_1\alpha_2 - \frac{1}{6}\right] \bar\Phi(x-\alpha),$$

$$x_2^2 = \sum_\alpha \left[\alpha_2^2 + \frac{1}{3}, \alpha_2^2 - \frac{1}{3}\right] \bar\Phi(x-\alpha), \quad x_1^3 = \sum_\alpha \left[\alpha_1^3 + \alpha_1, \alpha_1^3 - \alpha_1\right] \bar\Phi(x-\alpha),$$

$$x_1^2x_2 = \sum_\alpha \left[\alpha_1^2\alpha_2 + \frac{1}{3}(\alpha_1+\alpha_2), \alpha_1^2\alpha_2 - \frac{1}{3}(\alpha_1+\alpha_2)\right] \bar{\Phi}(x-\alpha),$$

$$x_1x_2^2 = \sum_\alpha \left[\alpha_1\alpha_2^2 + \frac{1}{3}(\alpha_1+\alpha_2), \alpha_1\alpha_2^2 - \frac{1}{3}(\alpha_1+\alpha_2)\right] \bar{\Phi}(x-\alpha),$$

$$x_2^3 = \sum_\alpha \left[\alpha_2^3 + \alpha_2, \alpha_2^3 - \alpha_2\right] \bar{\Phi}(x-\alpha).$$

Let us now define the following *differential quasi-interpolant* (DQI) :

$$Q^*f(x) := \sum_{\alpha\in\mathbb{Z}^2} \left[\mathcal{D}_1^*f(\alpha), \mathcal{D}_2^*f(\alpha)\right] \bar{\Phi}(x-\alpha),$$

where the coefficients are the differential operators

$$\mathcal{D}_1^* := I + \frac{1}{6}\mathcal{D}^*, \qquad \mathcal{D}_2^* := I - \frac{1}{6}\mathcal{D}^*, \quad \text{with } \mathcal{D}^* := \partial_1^2 + \partial_1\partial_2 + \partial_2^2.$$

Then, from the above results, it is not hard to deduce the following

Theorem 1 *The differential quasi-interpolant Q^* is exact in the space $\mathbb{P}_3$ of bivariate cubic polynomials.*

Proof. From the expressions of second degree monomials m_{20}, m_{11} and m_{02}, it is clear that the differential operators $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ must be of the form given above. Then, it is easy to compute e.g.

$$\mathcal{D}_1^*m_{30}(\alpha) = \alpha_1^3 + \frac{1}{6}[6\alpha_1], \quad \mathcal{D}_1^*m_{21}(\alpha) = \alpha_1^2\alpha_2 + \frac{1}{6}[2\alpha_2 + 2\alpha_1],$$

which coincide with the first components of the expansions (2) of the first two monomials of degree 3 in terms of translates of multi-box splines. The same proof holds for $\mathcal{D}_2^*$ and for the remaining monomials. ■

3.2 Differential quasi-interpolant of higher order

Assume we are working on the uniform grid $h\mathbb{Z}^2$ and that f is sufficiently smooth. Let

$$\sum_{\substack{k,l\ge 0\\ k+l\le m}} a_{k,l}x_1^kx_2^l$$

be its bivariate Taylor polynomial of degree $m \ge 5$. Using the formulas for $Qf(0,0)$, $\partial_1 Qf(0,0)$ and $\partial_2 Qf(0,0)$ at the end of Section 2, we evaluate Q_h^*f and its derivatives at the origin in terms of the Taylor coefficients of f, and get the following interesting expansions

$$(Q_h^*f - f)(0,0) = -\frac{h^4}{9}(2a_{4,0} + a_{3,1} + a_{2,2} + a_{1,3} + 2a_{0,4}) + O(h^6),$$

$$\partial_1(Q_h^*f - f)(0,0) = O(h^4), \quad \partial_2(Q_h^*f - f)(0,0) = O(h^4).$$

In the same way we can also show that

$$\partial_1^r\partial_2^s(Q_h^*f - f)(0,0) = O(h^2), \quad r+s = 2.$$

From that, we deduce the following approximation properties of Q_h^*:

1. The approximation order is 4 at the origin, and by translation, at any vertex of $h\mathbb{Z}^2$.
2. The approximation order of first derivatives at $h\mathbb{Z}^2$ is 4.
3. The approximation order of second derivatives at $h\mathbb{Z}^2$ is 2.

Moreover, the coefficient of h^4 in the expansion for $(Q_h^* f - f)(0,0)$ has a simple form and leads us to introduce the differential operator

$$\mathcal{D}^{\#} := (\mathcal{D}^*)^2 = (\partial_1^2 + \partial_1\partial_2 + \partial_2^2)^2 = \partial_1^4 + 2\partial_1^3\partial_2 + 3\partial_1^2\partial_2^2 + 2\partial_1\partial_2^3 + \partial_2^4,$$

and to define the *new differential quasi-interpolant*

$$Q^{\#} f := \sum_{\alpha} \left[\mathcal{D}_1^{\#} f(\alpha), \mathcal{D}_2^{\#} f(\alpha)\right] \bar{\Phi}(x-\alpha),$$

where the two differential operators are defined by

$$\mathcal{D}_1^{\#} := I + \frac{1}{6}\mathcal{D}^* + \frac{1}{108}\mathcal{D}^{\#}, \quad \mathcal{D}_2^{\#} := I - \frac{1}{6}\mathcal{D}^* + \frac{1}{108}\mathcal{D}^{\#}$$

In that case, we obtain the following expansions at the origin:

$$(Q_h^{\#} f - f)(0,0) = -\frac{h^6}{54}(8a_{6,0} + 4a_{5,1} + 2a_{4,2} + a_{3,3} + 2a_{2,4} + 4a_{1,5} + 8a_{0,6}) + O(h^8),$$

$$\partial_1(Q_h^{\#} f - f)(0,0) = -\frac{h^4}{9}(11a_{5,0} + 5a_{4,1} + 3a_{3,2} + a_{2,3} + a_{1,4}) + O(h^6),$$

$$\partial_2(Q_h^{\#} f - f)(0,0) = -\frac{h^4}{9}(11a_{0,5} + 5a_{1,4} + 3a_{2,3} + a_{3,2} + a_{4,1}) + O(h^6),$$

$$\partial_1^2(Q_h^{\#} f - f)(0,0) = -\frac{h^2}{6}(10a_{4,0} + 5a_{3,1} + 4a_{2,2} + 2a_{1,3} + 4a_{0,4}) + O(h^4),$$

$$\partial_1\partial_2(Q_h^{\#} f - f)(0,0) = -\frac{h^2}{3}(2a_{4,0} + a_{3,1} + 2a_{2,2} + a_{1,3} + 2a_{0,4}) + O(h^4),$$

$$\partial_2^2(Q_h^{\#} f - f)(0,0) = -\frac{h^2}{6}(4a_{4,0} + 2a_{3,1} + 4a_{2,2} + 5a_{1,3} + 10a_{0,4}) + O(h^4).$$

From that, we deduce:

1. The approximation order is now 6 at the origin, and by translation, at any vertex of $h\mathbb{Z}^2$. As the leading coefficient only involves partial derivatives of order 6, it is exact for $f \in \mathbb{P}_5$.
2. The approximation order of first derivatives is still 4. As the leading coefficients only involve partial derivatives of order 5, it is exact for $f \in \mathbb{P}_4$.
3. The approximation order of second derivatives is still 2.

In particular, $Q_h^{\#} f$ is an *Hermite interpolant* of $f \in \mathbb{P}_4$ and a *Lagrange interpolant* of $f \in \mathbb{P}_5$ at the vertices of the grid $h\mathbb{Z}^2$.

Remark. It is still possible to increase the local approximation order by correcting with a differential operator of order 6. However, its expression is more complicated and is not the third power of the operator $\mathcal{D}^*$.

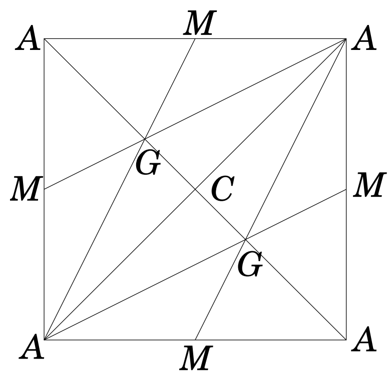

Figure 2: Specific points for superconvergence.

3.3 Superconvergence at some specific points

We consider the set of specific points (see Figure 2):

- A :=vertex of a square cell,
- M :=midpoint of an (horizontal or vertical) edge of a square.
- C :=center of a square,
- G :=center of a Powell-Sabin split.

We use the notations (sc) for *superconvergence* and

$$e_h^* := Q_h^* f - f, \quad e^{\#} := Q_h^{\#} f - f.$$

Then, using a computer algebra system (Maple), we get the following results for the *first differential quasi-interpolant* Q^* :

$$e_h^*(A) = O(h^4), \qquad \partial_1 e_h^*(A) = \partial_2 e_h^*(A) = O(h^4) \text{ (sc)}$$

$$e_h^*(M) = O(h^4), \qquad \partial_1 e_h^*(M) = \partial_2 e_h^*(M) = O(h^4), \text{ (sc)}$$

$$e_h^*(C) = O(h^4), \qquad \partial_1 e_h^*(C) = \partial_2 e_h^*(C) = O(h^4) \text{ (sc)}$$

$$e_h^*(G) = O(h^4), \qquad \partial_1 e_h^*(G) = \partial_2 e_h^*(G) = O(h^3).$$

In the same way, for the *second differential quasi-interpolant* $Q^{\#}$:

$$e_h^{\#}(A) = O(h^6), \text{ (sc)} \qquad \partial_1 e_h^*(A) = \partial_2 e_h^{\#}(A) = O(h^4) \text{ (sc)}$$

$$e_h^{\#}(M) = O(h^4), \qquad \partial_1 e_h^{\#}(M) = \partial_2 e_h^{\#}(M) = O(h^4), \text{ (sc)}$$

$$e_h^{\#}(C) = O(h^4), \qquad \partial_1 e_h^{\#}(C) = \partial_2 e_h^{\#}(C) = O(h^4) \text{ (sc)}$$

$$e_h^{\#}(G) = O(h^4), \qquad \partial_1 e_h^{\#}(G) = \partial_2 e_h^{\#}(G) = O(h^3).$$

Moreover, we have the following results about monomials in $\mathbb{P}_4$ and $\mathbb{P}_5$:

1. If $f \in \mathbb{P}_4$ is a monomial $x^m y^n$, $m+n=4$, then $\partial_1 Q^* f$ (resp. $\partial_2 Q^* f$) interpolates $\partial_1 f$ (resp. $\partial_2 f$) at the vertices A, midpoints M and center points C. However, $Q^* f$ does not interpolate f at any point of types A, M, C, G.

2. In addition, $Q^{\#} f$ interpolates f at vertices.

3. If $f \in \mathbb{P}_5$ is a monomial $x^m y^n$, $m+n=5$, then $Q^* f$ interpolates f at points A, M, C and $Q_h^* f - f = O(h^5)$ at points G (superconvergence). Moreover, there is still superconvergence of order 4 on first order partial derivatives.

4 Discrete quasi-interpolants in the plane

4.1 Discrete quasi-interpolants of the second order

The differential operators $\partial_1^2, \partial_1\partial_2, \partial_2^2$ can be substituted by discrete operators (finite differences) which coincide with them on cubic polynomials :

$$\partial_1^2 f(\alpha) \sim \delta_1^2 f(\alpha) := f(\alpha+e_1) - 2f(\alpha) + f(\alpha - e_1),$$

$$\partial_1\partial_2 f(\alpha) \sim \delta_1\delta_2 f(\alpha) := \frac{1}{4}[f(\alpha+e_3) - f(\alpha+e_4) - f(\alpha-e_4) + f(\alpha-e_3)],$$

$$\partial_2^2 f(\alpha) \sim \delta_2^2 f(\alpha) := f(\alpha+e_2) - 2f(\alpha) + f(\alpha - e_2).$$

Therefore the two discrete operators:

$$\mathcal{D}_1 := I + \frac{1}{6}\mathcal{D}, \qquad \mathcal{D}_2 := I - \frac{1}{6}\mathcal{D}, \quad \text{with } \mathcal{D} := \delta_1^2 + \delta_1\delta_2 + \delta_2^2,$$

also coincide respectively with $\mathcal{D}_1^*$ and $\mathcal{D}_2^*$ on the space $\mathbb{P}_3$. Their full expressions are respectively

$$\mathcal{D}_1 f(\alpha) := \frac{1}{3} f(\alpha) + \frac{1}{6}\left(f(\alpha \pm e_1) + f(\alpha \pm e_2)\right) + \frac{1}{24}\left(f(\alpha \pm e_3) - f(\alpha \pm e_4)\right),$$

$$\mathcal{D}_2 f(\alpha) := \frac{5}{3} f(\alpha) - \frac{1}{6}\left(f(\alpha \pm e_1) + f(\alpha \pm e_2)\right) - \frac{1}{24}\left(f(\alpha \pm e_3) - f(\alpha \pm e_4)\right).$$

The associated stencils are respectively :

−1/24	1/6	1/24		1/24	−1/6	−1/24
1/6	1/3	1/6		−1/6	5/3	−1/6
1/24	1/6	−1/24		−1/24	−1/6	1/24

One can now define the following discrete quasi-interpolant (dQI) :

$$\overline{Q}^* f(x) := \sum_{\alpha} \left[\mathcal{D}_1 f(\alpha), \mathcal{D}_2 f(\alpha)\right] \bar{\Phi}(x-\alpha).$$

and we immediately deduce a result similar to that of the preceding section:

Theorem 2 *The discrete quasi-interpolant $\overline{Q}^*$ is exact in the space $\mathbb{P}_3$ of bivariate cubic polynomials.*

4.2 Discrete quasi-interpolant of the fourth order

We construct new finite difference operators $\overline{\mathcal{D}}_1$ and $\overline{\mathcal{D}}_2$ approximating the two previous differential operators $\mathcal{D}_1^\#$ and $\mathcal{D}_2^\#$. Taking stencils with 19 points having an hexagonal shape a nd coefficients only depending on a sequence of 4 parameters $\mathbf{a} := [a_1; a_2; a_3; a_4]$:

$$\begin{array}{ccccc} & & a_3 & a_4 & a_3 \\ & a_4 & a_2 & a_2 & a_4 \\ a_3 & a_2 & a_1 & a_2 & a_3 \\ a_4 & a_2 & a_2 & a_4 & \\ a_3 & a_4 & a_3 & & \end{array}$$

we write that the two pairs of discrete and differential operators $(\overline{\mathcal{D}}_1, \mathcal{D}_1^\#)$ and $(\overline{\mathcal{D}}_2, \mathcal{D}_2^\#)$ coincide on the polynomial space $\mathbb{P}_4$: this will imply, thanks to the symmetries of coefficients, that they coincide on $\mathbb{P}_5$. The complete expression of $\overline{\mathcal{D}}_k f, k = 1, 2$ is then of the form

$$\begin{aligned} \overline{\mathcal{D}}_k f(\alpha) = a_1 f(\alpha) &+ a_2 \left(f(\alpha \pm e_1) + f(\alpha \pm e_2) + f(\alpha \pm e_3)\right) \\ &+ a_3 \left(f(\alpha \pm 2e_1) + f(\alpha \pm 2e_2) + f(\alpha \pm 2e_3)\right) \\ &+ a_4 \left(f(\alpha \pm e_4) + f(\alpha \pm e_5) + f(\alpha \pm e_6)\right). \end{aligned}$$

With the help of a computer algebra system, we find the two following sequences, each depending on two arbitrary parameters (ξ_i, η_i), $i = 1, 2$:

$$\mathbf{a}_1 := \left[\frac{4}{9} - \xi_1 - \frac{1}{18}\eta_1 ; \frac{7}{72} + \frac{1}{3}\xi_1 + \frac{1}{54}\eta_1 ; \frac{1}{6}\xi_1 + \frac{1}{108}\eta_1 ; -\frac{1}{216} - \frac{1}{3}\xi_1 - \frac{1}{54}\eta_1\right],$$

$$\mathbf{a}_2 := \left[\frac{16}{9} + \xi_2 - \frac{1}{18}\eta_2 ; -\frac{11}{72} - \frac{1}{3}\xi_2 + \frac{1}{54}\eta_2 ; -\frac{1}{6}\xi_2 + \frac{1}{108}\eta_2 ; \frac{5}{216} + \frac{1}{3}\xi_2 - \frac{1}{54}\eta_2\right].$$

One can now define the *new discrete quasi-interpolant*

$$\overline{Q}^\# f(x) := \sum_\alpha \left[\overline{\mathcal{D}}_1 f(\alpha), \overline{\mathcal{D}}_2 f(\alpha)\right] \bar{\Phi}(x - \alpha).$$

As in Section 3.2, given the function

$$f(x, y) = \sum_{k,l \geq 0} a_{k,l} x^k y^l$$

and computing the coefficients of $\overline{Q}f$ for indices in a neighbourhood of $\alpha = (0, 0)$, we get the following expansions which are close to those obtained for the higher order differential QI of Section 3.2:

$$\begin{aligned} (\overline{Q}_h^\# f - f)(0,0) &= O(h^6), \\ \partial_1(\overline{Q}_h^\# f - f)(0,0) &= -\frac{h^4}{9}\left(11a_{5,0} + 5a_{4,1} + 3a_{3,2} + a_{2,3} + a_{1,4}\right) + O(h^6), \\ \partial_2(\overline{Q}_h^\# f - f)(0,0) &= -\frac{h^4}{9}\left(11a_{0,5} + 5a_{1,4} + 3a_{2,3} + a_{3,2} + a_{4,1}\right) + O(h^6), \\ \partial_1^2(\overline{Q}_h^\# f - f)(0,0) &= -\frac{h^2}{6}\left(10a_{4,0} + 5a_{3,1} + 4a_{2,2} + 3a_{1,3} + 4a_{0,4}\right) + O(h^4), \end{aligned}$$

$$\partial_1\partial_2(\overline{Q}_h^{\#} f - f)(0,0) = -\frac{h^2}{3}\,(2a_{4,0} + a_{3,1} + 2a_{2,2} + a_{1,3} + 2a_{0,4}) + O(h^4),$$

$$\partial_2^2(\overline{Q}_h^{\#} f - f)(0,0) = -\frac{h^2}{6}\,(4a_{4,0} + 2a_{3,1} + 4a_{2,2} + 5a_{1,3} + 10a_{0,4}) + O(h^4).$$

Therefore, we can conclude, as in Section 3.2, that for this dQI :

1. The approximation order is now 6 at the origin, and by translation, at any vertex of $h\mathbb{Z}^2$. As the leading coefficient only involves partial derivatives of order 6, it is exact for $f \in \mathbb{P}_5$.
2. The approximation order of first derivatives is still 4. As the leading coefficients only involve partial derivatives of order 5, it is exact for $f \in \mathbb{P}_4$.
3. The approximation order of second derivatives is still 2.

In particular, $\overline{Q}_h^{\#} f$ is an *Hermite interpolant* of $f \in \mathbb{P}_4$ and a *Lagrange interpolant* of $f \in \mathbb{P}_5$ at the vertices of the grid $h\mathbb{Z}^2$.

4.3 Superconvergence properties of dQIs

As for differential QIs (Section 3.3), we consider the same set of specific points of type A,B,C and D depicted in Figure 2.

We use the notations (sc) for *superconvergence* and

$$e_h := \overline{Q}_h^{*} f - f, \quad \bar{e}_h := \overline{Q}_h^{\#} f - f.$$

Then, using a computer algebra system (Maple), we get the following results for the *first discrete quasi-interpolant* $\overline{Q}^*$:

$$e_h(A) = O(h^4), \qquad \partial_1 e_h(A) = \partial_2 e_h(A) = O(h^4) \text{ (sc)}$$

$$e_h(M) = O(h^4), \qquad \partial_1 e(M) = \partial_2 e_h(M) = O(h^4), \text{ (sc)}$$

$$e_h(C) = O(h^4), \qquad \partial_1 e_h(C) = \partial_2 e_h(C) = O(h^4) \text{ (sc)}$$

$$e_h(G) = O(h^4), \qquad \partial_1 e_h(G) = \partial_2 e_h(G) = O(h^3).$$

In the same way, for the *second discrete quasi-interpolant* $\overline{Q}^{\#}$:

$$\bar{e}_h(A) = O(h^6), \text{ (sc)} \qquad \partial_1 \bar{e}(A) = \partial_2 \bar{e}_h(A) = O(h^4) \text{ (sc)}$$

$$\bar{e}_h(M) = O(h^4), \qquad \partial_1 \bar{e}_h(M) = \partial_2 \bar{e}_h(M) = O(h^4), \text{ (sc)}$$

$$\bar{e}_h(C) = O(h^4), \qquad \partial_1 \bar{e}_h(C) = \partial_2 \bar{e}_h(C) = O(h^4) \text{ (sc)}$$

$$\bar{e}_h(G) = O(h^4), \qquad \partial_1 \bar{e}_h(G) = \partial_2 \bar{e}_h(G) = O(h^3).$$

Moreover, we have the following results about polynomials in $\mathbb{P}_4$ and $\mathbb{P}_5$:

1. If $f \in \mathbb{P}_4$ is a polynomial, then $\partial_1 \overline{Q}^* f$ (resp. $\partial_2 \overline{Q}^* f$) interpolates $\partial_1 f$ (resp. $\partial_2 f$) at the vertices A, midpoints M and center points C. However, $\overline{Q}^* f$ does not interpolate f at any point of types A, M, C, G. $\overline{Q}^{\#} f$ has the same properties, however it interpolates f at vertices.

2. If $f \in \mathbb{P}_5$ is a polynomial, then $\overline{Q}^* f$ interpolates f at points A, M, C and $\overline{Q}_h^* f - f = O(h^5)$ at points G (superconvergence). Moreover, there is still superconvergence of order 4 on first order partial derivatives. $\overline{Q}^{\#} f$ has the same properties.

5 Norm and error estimates for the discrete quasi-interpolants

5.1 Infinity norms of discrete quasi-interpolants

For bounded functions f, a first upper bound on the maximum norm of $\overline{Q}^*$ can be obtained by bounding above its coefficients:

$$|\mathcal{D}_1 f(\alpha)| \leq \frac{7}{6}\|f\|_\infty, \quad |\mathcal{D}_2 f(\alpha)| \leq \frac{5}{2}\|f\|_\infty,$$

therefore, as the sum of integer translates of generators φ_1 and φ_2 is equal to one, we immediately deduce

$$\|\overline{Q}^*\|_\infty \leq \frac{5}{2}.$$

More precisely, let us define the following *quasi-Lagrange spline* (called *superfunction* elsewhere [5, 20]) :

$$\begin{aligned}\psi(x) := &\frac{1}{3}\varphi_1(x) + \frac{1}{6}\left(\varphi_1(x+e_1) + \varphi_1(x+e_2)\right) + \frac{1}{6}\left(\varphi_1(x-e_1) + \varphi_1(x-e_2)\right)\\ &+\frac{1}{24}\left(\varphi_1(x+e_3) - \varphi_1(x+e_4)\right) + \frac{1}{24}\left(\varphi_1(x-e_3) - \varphi_1(x-e_4)\right)\\ &+\frac{5}{3}\varphi_2(x) - \frac{1}{6}\left(\varphi_2(x+e_1) + \varphi_2(x+e_2)\right) - \frac{1}{6}\left(\varphi_2(x-e_1) + \varphi_2(x-e_2)\right)\\ &-\frac{1}{24}\left(\varphi_2(x+e_3) - \varphi_2(x+e_4)\right) - \frac{1}{24}\left(\varphi_2(x-e_3) - \varphi_2(x-e_4)\right).\end{aligned}$$

Then the dQI can be written in the simple *quasi-Lagrange* form:

$$\overline{Q}^* f(x) = \sum_{\alpha\in\mathbb{Z}^2} f(\alpha)\psi(x-\alpha).$$

The infinite norm of $\overline{Q}^*$ is then equal to the max-norm of its Lebesgue function Λ:

$$\|\overline{Q}^*\|_\infty = |\Lambda|_\infty, \quad \text{with} \quad \Lambda(x) := \sum_\alpha |\psi(x-\alpha)|.$$

Remark: The computation of a better upper bound on $\|\overline{Q}^*\|_\infty$ is based upon a good estimate of $|\Lambda|_\infty$. The unit square $\Omega := [0,1]^2$ is covered by the supports of 34 integer translates of the quasi-Lagrange spline. Because of the symmetries of the support of ψ, it is enough to study the maximum of Λ on a subdomain Ω^* of Ω. As the explicit BB(=Bernstein-Bézier) form of each cubic polynomial piece p_α of $\psi(x-\alpha)$ is known on Ω^*, we can bound $|p_\alpha|$ by the polynomial p_α^* having as BB-coefficients the absolute values of those of p_α. Then it follows that Λ does not exceed the sum of polynomials p_α^*. This computation will be detailed elsewhere.

Infinity norm of the 4th order dQI

We have already seen that $\|\overline{Q}^*\|_\infty \le 2.5$. Here, choosing $(\xi_1,\eta_1)=(0,-\frac{1}{4})$ and $(\xi_2,\eta_2)=(0,\frac{5}{4})$, we obtain respectively:

$$\mathbf{a}_1 := \left[\frac{11}{24};\frac{5}{54};-\frac{1}{432};0\right], \qquad \mathbf{a}_2 := \left[\frac{41}{24};-\frac{7}{54};\frac{5}{432};0\right]$$

from which we deduce

$$|\overline{\mathcal{D}}_1|_\infty = \frac{37}{36}, \qquad |\overline{\mathcal{D}}_2|_\infty = \frac{23}{9},$$

and finally $\|\overline{Q}^\#\|_\infty \le \frac{23}{9} \approx 2.55$.

5.2 Error estimates

We can use the above bounds on the norms of our dQI to investigate their approximation order.

Theorem 3 *The discrete quasi-interpolants $\overline{Q}^*$ and $\overline{Q}^\#$ possess full approximation order in the sense of (1).*

Proof. Let $Q=\overline{Q}^*$ or $\overline{Q}^\#$. Denote by $d_{\infty,\tau}(f,\mathbb{P}_3)$ the best approximation of f from $\mathbb{P}_3$ on a triangle τ in L_∞-norm. According to the result of Section 5.1 on the norms of $\overline{Q}^*,\overline{Q}^\#$ and classical results in approximation theory, we obtain, in view of the exactness of Q on $\mathbb{P}_3$,

$$\|f-Q_hf\|_{\infty,\tau} \le (1+\|Q_h\|_\infty)\, d_{\infty,\tau}(f,\mathbb{P}_3) \le K_1\, d_{\infty,\tau}(f,\mathbb{P}_3), \quad \text{for all triangles of } hV,$$

where $0<K_1<7/2$. Now, if f has bounded fourth order partial derivatives, then it is well known that $d_{\infty,\tau}(f,\mathbb{P}_3)\le K_2h^4\|D^4f\|_\infty$, whence (1) for $\gamma=0$. The case $1\le|\gamma|\le 3$ is obtained in the same way. ■

6 Quasi-interpolants in a rectangular domain

Let $R_{m,n}$ be the rectangular domain $[0,m]\times[0,n]$ and let $\mathcal{S}_{m,n}$ be the space $\mathcal{S}_3^2(R_{m,n})$ of C^2 cubic splines on $R_{m,n}$ endowed with the 6-direction mesh. According to [8], the dimension of this space is

$$d_{m,n} := \dim(\mathcal{S}_{m,n}) = 2mn+4(m+n)+6.$$

There are $(m+1)(n+1)$ shifts of φ_1 and $(m+3)(n+3)-2$ shifts of φ_2 whose supports overlap with the domain, totalling to

$$b_{m,n} := 2mn + 4(m+n) + 8 = d_{m,n} + 2.$$

Therefore, the generators of $\mathcal{S}_{m,n}$ are *linearly dependent*. However, this fact is not important for quasi-interpolation.

On the other hand, the coefficient functionals of boundary generators use data points which lie outside the domain, thus we are led to modify these functionals in order to *use only data points that lie inside* $R_{m,n}$ (or on its boundary) and still give a quasi-interpolant exact on cubic polynomials. Let us give some details for generators whose centers of supports are close to the origin or to the boundary OA with $A=(m,0)$.The other cases can be treated similarly.

Example 1. Consider the multi-box spline φ_1 whose support is centered at the origin. Its coefficient in the differential quasi-interpolant Q^*f is equal to

$$\mathcal{D}_1^* f(0,0) = (f + \frac{1}{6}\mathcal{D}^* f)(0,0).$$

The 10-dimensional vector b of the values $\mathcal{D}_1^* f(0,0)$ on all monomials $f \in \mathbb{P}_3$ (in lexicographic order) is as follows

$$b = [1,0,0,1/3,1/6,1/3,0,0,0,0]^T.$$

In order to approximate $\mathcal{D}_1^* f$ by a finite difference operator based on values of f at gridpoints close to the origin, we choose the discrete operator

$$\begin{aligned}\mathcal{D}_1 f(0,0) := a_1 f(0,0) + a_2(f(1,0)+f(0,1)) + a_3(f(2,0)+f(0,2)) + a_4 f(1,1)\\ + a_5(f(3,0)) + f(0,3)) + a_6(f(2,1)+f(1,2)),\end{aligned}$$

and require that its values on monomials of $\mathbb{P}_3$ coincide with the components of b. Then, the coefficients a_i satisfy the following six equations

$$\begin{aligned}
a_1 + 2a_2 + 2a_3 + a_4 + 2a_5 + 2a_6 &= 1\\
a_2 + 2a_3 + a_4 + 3a_5 + 3a_6 &= 0\\
a_2 + 4a_3 + a_4 + 9a_5 + 5a_6 &= 1/3\\
a_4 + 4a_6 &= 1/6\\
a_2 + 8a_3 + a_4 + 27a_5 + 9a_6 &= 0\\
a_4 + 6a_6 &= 0
\end{aligned}$$

the unique solution of which is

$$a = [2, -5/4, 3/4, 1/2, -1/6, -1/12]^T.$$

Therefore, we can choose as coefficient of φ_1 in the discrete quasi-interpolant Qf the finite difference

$$\mathcal{D}_1 f(0,0) = 2f(0,0) - \frac{5}{4}(f(1,0)+f(0,1)) + \frac{3}{4}(f(2,0)+f(0,2)) + \frac{1}{2}f(1,1)$$

$$-\frac{1}{6}(f(3,0)+f(0,3))-\frac{1}{12}(f(2,1)+f(1,2)).$$

We see that $|\mathcal{D}_1|_\infty = 7$, a rather high value.

Example 2. Let us now consider the multi-box spline φ_2 whose support is centered at the point $(-1,-1)$. Its coefficient in the differential quasi-interpolant Q^*f is equal to

$$\mathcal{D}_2^* f(-1,-1) = (f - \frac{1}{6}\mathcal{D}^* f)(-1,-1).$$

The 10-dimensional vector b of the values $\mathcal{D}_2^* f(0,0)$ on all monomials $f \in \mathbb{P}_3$ is

$$b = [1,-1,-1,2/3,5/6,2/3,0,-1/3,-1/3,0]^T.$$

We approximate $\mathcal{D}_2^* f$ by a finite difference operator $\mathcal{D}_2 f$ of the same type as $\mathcal{D}_1 f$. Solving the associate system of linear equations gives

$$a = [8,-29/4,41/12,19/6,-2/3,-7/12],$$

therefore, we can choose as coefficient of φ_2 in the discrete quasi-interpolant Qf the following finite difference

$$\mathcal{D}_2 f(0,0) = 8f(0,0) - \frac{29}{4}(f(1,0)+f(0,1)) + \frac{41}{12}(f(2,0)+f(0,2))$$

$$+\frac{19}{6}f(1,1) - \frac{2}{3}(f(3,0)+f(0,3)) - \frac{7}{12}(f(2,1)+f(1,2)).$$

We see that $|\mathcal{D}_2|_\infty = 35$, a very high value.

Example 3. Consider the multi-box spline of type φ_1 whose support is centered at the boundary point $(1,0)$. The coefficient $\mathcal{D}_1^* f(1,0)$ is approximated a discrete operator based on the same data points as in the previous examples :

$$\mathcal{D}_1 f(1,0) := a_1 f(0,0) + a_2 f(1,0) + a_3 f(0,1) + a_4 f(2,0) + a_5 f(1,1) + a_6 f(0,2)$$

$$+a_7 f(3,0) + a_8 f(2,1) + a_9 f(1,2) + a_{10} f(0,3).$$

However, the vector b of values of $\mathcal{D}_1^* f(1,0)$ on monomials of $\mathbb{P}_3$ is equal to

$$b = [1,1,0,4/3,1/6,1/3,2,1/3,1/3,0]^T,$$

therefore the solution of the linear system is

$$a = [1/2,3/4,-3/4,1/12,-1/6,7/12,0,1/12,1/12,-1/6]$$

It corresponds to the following discrete operator:

$$\mathcal{D}_1 f(1,0) = \frac{1}{2}f(0,0) + \frac{3}{4}(f(1,0)-f(0,1)) + \frac{1}{12}(f(2,0)+7f(0,2))$$

$$-\frac{1}{6}f(1,1) + \frac{1}{12}(f(2,1)+f(1,2)-2f(0,3)).$$

In that case, we get $|\mathcal{D}_1|_\infty = 19/6 \approx 3.16$, a lower value than in the first example.

Example 4. As a last example, we consider the multi-box spline of type φ_2 whose support is centered at the boundary point $(1,-1)$. The coefficient $\mathcal{D}_1^* f(1,-1)$ is approximated by the same discrete operator as in the third example :

$$\mathcal{D}_2 f(1,-1) := a_1 f(0,0) + a_2 f(1,0) + a_3 f(0,1) + a_4 f(2,0) + a_5 f(1,1) + a_6 f(0,2)$$
$$+ a_7 f(3,0) + a_8 f(2,1) + a_9 f(1,2) + a_{10} f(0,3).$$

Here, the vector b of values of $\mathcal{D}_1^* f(1,-1)$ on monomials of $\mathbb{P}_3$ is equal to

$$b = [1, 1, -1, 2/3, -7/6, 2/3, 0, -1, 1, 0]^T,$$

therefore the solution of the linear system is now

$$a = [0, 15/4, -5/4, -1/4, -7/2, 7/4, 0, 1/12, 13/12, -2/3]^T.$$

It corresponds to the following discrete operator:

$$\mathcal{D}_2 f(1,0) = \frac{5}{4}(3f(1,0) - f(0,1)) - \frac{1}{4}(f(2,0) + 14f(1,1) - 7f(0,2))$$
$$+ \frac{1}{12}(f(2,1) + 13f(1,2) - 8f(0,3)),$$

for which we get $|\mathcal{D}_2|_\infty = 37/3 \approx 12.3$, a lower value than in the second example.

We thus observe that functionals associated with some boundary multi-box splines can have infinite norms with high values. This can increase the value of the infinite norm of the quasi-interpolant and eventually give some trouble in numerical computations. For that reason, we shall develop in a further paper alternative functionals having larger supports and lower infinite norms (as is done in the corresponding work by Sara Remogna [26] on quadratic spline QIs).

7 Numerical examples

In this section we present numerical results confirming good approximation properties of the proposed quasi-interpolants $Q = Q^*, Q^{\#}, \overline{Q}^*, \overline{Q}^{\#}$.

We have used the well-known Franke test function [14] given by

$$\begin{aligned} f(x,y) &= \frac{3}{4}\exp\Big[-\frac{(9x-2)^2+(9y-2)^2}{4}\Big] + \frac{3}{4}\exp\Big[-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10}\Big] \\ &+ \frac{1}{2}\exp\Big[-\frac{(9x-7)^2+(9y-3)^2}{4}\Big] - \frac{1}{5}\exp\Big[-(9x-4)^2-(9y-7)^2\Big]. \end{aligned}$$

Quasi-interpolants $Q_h f$ using the values of f (and its derivatives in the case of DQIs $Q^*, Q^{\#}$) were computed for $h = 1/n$, with $n = 16, 32, \ldots, 512$. The resulting C^2 cubic spline and its gradient and Hessian were evaluated on a dense grid in the square domain $\Omega = [0,1]^2$ and compared with the corresponding values of f, ∇f and Hf. The errors are presented in Tables 1, 3 and 5, respectively. They all confirm as expected the full approximation order (h^4 for the function values, h^3 for the gradients and h^2 for the Hessian) of all four quasi-interpolants, where the errors for the higher order quasi-interpolants $Q^{\#}, \overline{Q}^{\#}$ tend to be better than $Q^*, \overline{Q}^*$. It is interesting to note that discrete QI $\overline{Q}^{\#}$ often significantly outperforms differential QI Q^* and is rarely worse than $Q^{\#}$.

It was shown in Section 4.3 that $f - Q_h^{\#} f$ and $f - \overline{Q}_h^{\#} f$ are *superconvergent* at the vertices of the square cells, that is at the points in $h\mathbb{Z}^2$. Indeed, the approximation order h^6 in this situation is confirmed by the numerical results presented in Table 2. Moreover, summarising the other results of Sections 3.3 and 4.3, we see that the gradients of all four quasi-interpolants are superconvergent at the points of type A, M and C, that is at the grid $\frac{h}{2}\mathbb{Z}^2$. This observation is confirmed by the numerical results in Table 4 showing the approximation order h^4 for the gradients at these points.

	DQI Q^*		dQI $\overline{Q}^*$		DQI $Q^{\#}$		dQI $\overline{Q}^{\#}$	
n	f error	ratio	f error	ratio	f error	ratio	f error	ratio
16	7.08e-03		9.50e-03		3.98e-03		6.57e-03	
32	5.90e-04	12.0	8.86e-04	10.7	2.44e-04	16.3	3.70e-04	17.7
64	3.95e-05	14.9	6.14e-05	14.4	1.47e-05	16.6	1.74e-05	21.3
128	2.51e-06	15.8	3.93e-06	15.6	9.06e-07	16.2	9.50e-07	18.3
256	1.58e-07	15.9	2.48e-07	15.9	5.65e-08	16.0	5.72e-08	16.6
512	9.86e-09	16.0	1.55e-08	16.0	3.53e-09	16.0	3.54e-09	16.2

Table 1: Maximum function error $\|f - Q_h f\|_\infty$, $h = 1/n$, and ratio $\frac{\|f-Q_{2h}f\|_\infty}{\|f-Q_h f\|_\infty}$ for the four types of quasi-interpolants, where $\|f\|_\infty := \|f\|_{L_\infty(\Omega)}$.

	DQI $Q^{\#}$		dQI $\overline{Q}^{\#}$	
n	f error	ratio	f error	ratio
16	7.00e-04		1.96e-03	
32	1.10e-05	63.9	1.04e-04	18.9
64	1.39e-07	78.8	2.18e-06	47.6
128	1.86e-09	74.7	3.70e-08	58.9
256	4.01e-11	46.4	5.93e-10	62.4
512	2.46e-12	16.3	9.31e-12	63.7

Table 2: Superconvergence at grid points $h\mathbb{Z}^2$. Maximum function error $\|f - Q_h f\|_\infty$, $h = 1/n$, and ratio $\frac{\|f-Q_{2h}f\|_\infty}{\|f-Q_h f\|_\infty}$ for the higher order quasi-interpolants $Q^{\#}, \overline{Q}^{\#}$, where $\|f\|_\infty := \|f\|_{\ell_\infty(\Omega \cap h\mathbb{Z}^2)}$. The deterioration of the approximation order of $Q^{\#}$ for $n = 256$ and 512 is most probably caused by rounding errors as the accuracy approaches the unit round-off.

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n	DQI Q^* ∇f error	ratio	dQI $\overline{Q}^*$ ∇f error	ratio	DQI $Q^\#$ ∇f error	ratio	dQI $\overline{Q}^\#$ ∇f error	ratio
16	5.64e-02		7.11e-02		6.08e-02		6.27e-02	
32	7.03e-03	8.0	9.42e-03	7.5	7.12e-03	8.5	7.46e-03	8.4
64	8.50e-04	8.3	1.18e-03	8.0	8.50e-04	8.4	8.70e-04	8.6
128	1.05e-04	8.1	1.47e-04	8.0	1.05e-04	8.1	1.05e-04	8.2
256	1.31e-05	8.0	1.84e-05	8.0	1.31e-05	8.0	1.31e-05	8.0
512	1.63e-06	8.0	2.30e-06	8.0	1.63e-06	8.0	1.63e-06	8.0

Table 3: Relative maximum gradient error $\|\nabla(f - Q_h f)\|_\infty / \|\nabla f\|_\infty$, $h = 1/n$, and ratio $\frac{\|\nabla(f-Q_{2h}f)\|_\infty}{\|\nabla(f-Q_h f)\|_\infty}$ for the four types of quasi-interpolants, where $\|\nabla f\|_\infty := \|\sqrt{\partial_1 f^2 + \partial_2 f^2}\|_{L_\infty(\Omega)}$.

n	DQI Q^* ∇f error	ratio	dQI $\overline{Q}^*$ ∇f error	ratio	DQI $Q^\#$ ∇f error	ratio	dQI $\overline{Q}^\#$ ∇f error	ratio
16	5.27e-02		6.88e-02		4.51e-02		5.35e-02	
32	4.69e-03	11.2	6.99e-03	9.8	2.89e-03	15.6	3.63e-03	14.7
64	3.18e-04	14.7	4.98e-04	14.0	1.80e-04	16.0	1.94e-04	18.7
128	2.04e-05	15.6	3.23e-05	15.4	1.13e-05	16.0	1.15e-05	16.9
256	1.28e-06	15.9	2.04e-06	15.8	7.02e-07	16.0	7.06e-07	16.3
512	8.04e-08	16.0	1.28e-07	16.0	4.39e-08	16.0	4.40e-08	16.1

Table 4: Gradient superconvergence at half-grid points $\frac{h}{2}\mathbb{Z}^2$. Relative maximum gradient error $\|\nabla(f - Q_h f)\|_\infty / \|\nabla f\|_\infty$, $h = 1/n$, and ratio $\frac{\|\nabla(f-Q_{2h}f)\|_\infty}{\|\nabla(f-Q_h f)\|_\infty}$ for the four types of quasi-interpolants, where $\|\nabla f\|_\infty := \|\sqrt{\partial_1 f^2 + \partial_2 f^2}\|_{\ell_\infty(\Omega \cap \frac{h}{2}\mathbb{Z}^2)}$.

n	DQI Q^* Hf error	ratio	dQI $\overline{Q}^*$ Hf error	ratio	DQI $Q^\#$ Hf error	ratio	dQI $\overline{Q}^\#$ Hf error	ratio
16	3.18e-01		3.93e-01		4.33e-01		3.78e-01	
32	1.31e-01	2.4	1.70e-01	2.3	1.45e-01	3.0	1.50e-01	2.5
64	3.55e-02	3.7	4.80e-02	3.5	3.64e-02	4.0	3.71e-02	4.0
128	9.10e-03	3.9	1.25e-02	3.9	9.16e-03	4.0	9.22e-03	4.0
256	2.29e-03	4.0	3.15e-03	4.0	2.30e-03	4.0	2.30e-03	4.0
512	5.74e-04	4.0	7.89e-04	4.0	5.75e-04	4.0	5.75e-04	4.0

Table 5: Relative maximum Hessian error $\|H(f - Q_h f)\|_\infty / \|Hf\|_\infty$, $h = 1/n$, and ratio $\frac{\|H(f-Q_{2h}f)\|_\infty}{\|H(f-Q_h f)\|_\infty}$ for the four types of quasi-interpolants, where $\|Hf\|_\infty := \|\sqrt{(\partial_1^2 f)^2 + (\partial_1\partial_2 f)^2 + (\partial_2^2 f)^2}\|_{L_\infty(\Omega)}$.

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