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STABLE APPROXIMATION AND INTERPOLATION WITH C^1 QUARTIC BIVARIATE SPLINES *

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Abstract. We show how two recent algorithms [6,7,14] for computing C^1 quartic interpolating splines can be stabilized to insure that for smooth functions, they provide full approximation power with approximation constants depending only on the smallest angle in the triangulation.

Key words. bivariate splines, scattered data interpolation, stable approximation

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1. Introduction. Suppose $\mathcal{V} := \{v_i\}_{i=1}^n$ is a set of points in the plane. We are interested in the following

PROBLEM 1.1. *Find a triangulation $\triangle$ whose vertices contain the points of $\mathcal{V}$, and an operator Q mapping smooth functions into $\mathcal{S}_4^1(\triangle)$ so that*

$$Qf(v_i) = f(v_i), \qquad i = 1, \ldots, n, \tag{1.1}$$

and Q provides optimal order approximation in the sense that if f is in the Sobolev space $W_\infty^5(\Omega)$, then

$$\|f - Qf\|_\infty \le K|\triangle|^5 |f|_{5,\infty}. \tag{1.2}$$

Here $|\triangle|$ is the diameter of the largest triangle in $\triangle$, $|\cdot|_{5,\infty}$ is the usual Sobolev semi-norm, K is a constant depending only on the smallest angle $\theta_\triangle$ in $\triangle$, and $\mathcal{S}_d^r(\triangle)$ is the space of bivariate splines of smoothness r and degree d defined on the union Ω of the triangles in $\triangle$.

If we do not insist on full approximation power, then the problem was solved already in [3], where it was shown how to construct interpolating splines in $\mathcal{S}_4^1(\triangle)$ for *arbitrary* triangulations $\triangle$ of the set $\mathcal{V}$. On the other hand, it is well known [5] that $\mathcal{S}_4^1(\triangle)$ does not have full approximation power for general triangulations. So choosing the triangulation is a nontrivial matter, especially if we want an interpolation scheme where the constant K in (1.2) depends only on the smallest angle $\theta_\triangle$ and is otherwise independent of the geometry of the triangulation $\triangle$.

It is well known that great care is required in establishing approximation results for bivariate splines with constants that depend only on the degree of the splines and the smallest angle in the triangulation; cf. [8,11,15].

Problem 1.1 was addressed in [6,7,14], where algorithms for converting any given triangulation into a so-called type-O triangulation are presented, and an interpolation operator Q satisfying (1.1) and (1.2) was constructed, with constant K independent of f and $|\triangle|$. However, it appears to us that to make their operator Q satisfy (1.2)

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with a constant depending only on the smallest angle $\theta_\triangle$, one needs a stable local basis $\mathcal{B} := \{B_\xi\}_{\xi\in\mathcal{M}}$ for $\mathcal{S}_4^1(\triangle)$ on arbitrary type-O triangulations. By this we mean (cf. [8,12,15]) a basis such that

$$\operatorname{supp}(B_\xi) \subseteq \operatorname{star}^\ell(v_\xi) \text{ for some vertex } v_\xi, \qquad \text{all } \xi \in \mathcal{M}, \tag{1.3}$$

$$K_1\|c\|_\infty \;\le\; \|\sum_{\xi\in\mathcal{M}} c_\xi B_\xi\|_\infty \;\le\; K_2\|c\|_\infty, \tag{1.4}$$

where the integer ℓ and constants $K_1, K_2 > 0$ depend only on the smallest angle $\theta_\triangle$ in $\triangle$. Here $\operatorname{star}(v) = \operatorname{star}^1(v)$ is defined to be the union of the triangles with vertex at v, and $\operatorname{star}^\ell(v)$, $\ell \ge 2$, is defined recursively as the union of the stars of the vertices in $\operatorname{star}^{\ell-1}(v)$.

It is not difficult to see that the construction of local bases in [6,7] does not guarantee stability, see Remark 14.1. Moreover, the swapping algorithm in [7] does not control the size of the smallest angle in the resulting type-O triangulation; see Example 8.1 below. The purpose of this paper is to improve the construction by working with a more restricted class of triangulations which we call type-O_θ triangulations, modifying the construction of local bases to guarantee stability, and restructuring the swapping algorithm of [7] to control angles.

The paper is organized as follows. In Sect. 2 we present some standard Bernstein-Bézier notation, and in Sect. 3 review the concept of minimal determining sets. Type-O_θ triangulations are introduced in Sect. 4, while in Sect. 5 we discuss constructing minimal determining sets on disks for C^1 quartic splines. These results are then used in Sect. 6 to construct stable local bases for $\mathcal{S}_4^1(\triangle)$ on type-O_θ triangulations. In Sect. 7 we present an algorithm based on Clough-Tocher refinement for converting an arbitrary triangulation with smallest angle θ into a type-$\mathrm{O}_{\theta/2}$ triangulation. Sects. 8–10 deal with edge swapping, cells, and certain special vertices, and in Sect. 11 we present an algorithm based on swapping for converting an arbitrary triangulation with smallest angle θ into a type-$\mathrm{O}_{\kappa\theta}$ triangulation with an appropriate κ depending only on θ. In Sects. 12 and 13 we discuss quasi-interpolation, and Lagrange and Hermite interpolation. We conclude with several remarks in Sect. 14.

2. Notation. We make use of standard Bernstein-Bézier techniques for dealing with polynomial splines on a triangulation. In particular, given a polynomial p of degree d on a triangle $T := \langle u, v, w\rangle$, we use the Bernstein-Bézier representation

$$p = \sum_{i+j+k=d} c_{ijk}^T B_{ijk}^d,$$

where B_{ijk}^d are the Bernstein polynomials of degree d associated with T. It is standard practice to associate the coefficients c_{ijk}^T with the domain points in $\mathcal{D}_{d,T} := \{\xi_{ijk}^T := (iu + jv + kw)/d\}_{i+j+k=d}$. The set $R_\ell^T(u) := \{\xi_{d-\ell,j,\ell-j}^T\}_{j=0}^\ell$ is called the ℓ-th ring around u, while $D_\ell^T(u) := \bigcup_{m=0}^\ell R_m^T(u)$ is called the ℓ-th disk around u.

We shall also make extensive use of the standard smoothness conditions for piecewise polynomial functions. Suppose that $T := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T} := \langle u_4, u_3, u_2\rangle$ are two adjoining triangles that share the edge $e := \langle u_2, u_3\rangle$. Let p and $\tilde{p}$ be two polynomials of degree d with B-coefficients c_{ijk} and $\tilde{c}_{ijk}$ relative to T and $\widetilde{T}$, respectively.

Then it is well known that p and $\tilde{p}$ join with C^0 continuity across the edge e if and only if

$$\tilde{c}_{0,m,d-m} = c_{0,d-m,m}, \qquad m = 0, \ldots, d, \tag{2.1}$$

and that they join with C^1 continuity if and only if in addition

$$\tilde{c}_{1,m-1,d-m} = \alpha\, c_{1,d-m,m-1} + \beta\, c_{0,d-m+1,m-1} + \gamma\, c_{0,d-m,m}, \qquad m = 1, \ldots, d, \tag{2.2}$$

where (α, β, γ) are the barycentric coordinates of u_4 relative to the triangle T.

Given a triangulation $\triangle$, let $\mathcal{D}_{d,\triangle}$ be the union of $\mathcal{D}_{d,T}$ over all $T \in \triangle$. Then it is well known that there is a 1-1 correspondence between the space $\mathcal{S}_d^0(\triangle)$ and the set $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}}$, whereby the coefficients $\{c_\xi\}_{\xi\in\mathcal{D}_{d,\triangle}\cap T}$ are the B-coefficients of the polynomial $s|_T$.

As usual, we define $R_\ell(v)$ and $D_\ell(v)$ to be the unions of $R_\ell^T(v)$ and $D_\ell^T(v)$, respectively, over all triangles T attached to the vertex v.

3. Minimal determining sets. We recall [4] that if $\mathcal{S}$ is a linear subspace of $\mathcal{S}_d^0(\triangle)$, then $\mathcal{M} \subseteq \mathcal{D}_{d,\triangle}$ is said to be a determining set for $\mathcal{S}$ provided

$$\lambda_\xi s = 0 \quad \text{for all } \xi \in \mathcal{M} \quad \text{implies} \quad s \equiv 0,$$

where λ_ξ is the linear functional defined on $\mathcal{S}_d^0(\triangle)$ that picks off the B-coefficient c_ξ. $\mathcal{M}$ is called a minimal determining set (MDS) if there is no smaller determining set. It is known [4] that if $\mathcal{M}$ is a minimal determining set for $\mathcal{S}$, then $\dim \mathcal{S} = \#\mathcal{M}$.

If $\mathcal{M}$ is a MDS for $\mathcal{S}$, then for each $\xi \in \mathcal{M}$ there exists a unique spline $B_\xi \in \mathcal{S}$ satisfying

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}. \tag{3.1}$$

The splines B_ξ obviously form a basis for $\mathcal{S}$, commonly called the dual basis corresponding to $\mathcal{M}$.

DEFINITION 3.1 [12]. *A minimal determining set $\mathcal{M}$ for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ is called a* stable local MDS *provided that the corresponding dual basis $\mathcal{B} := \{B_\xi\}_{\xi\in\mathcal{M}}$ satisfies (1.3), and*

$$\|B_\xi\|_\infty \le K \tag{3.2}$$

for all $\xi \in \mathcal{M}$, where the constant K depends only on d and the smallest angle $\theta_\triangle$ in $\triangle$.

It was shown in [12] that if $\mathcal{M}$ is a stable local minimal determining set for a spline space $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$, then the dual basis $\mathcal{B}$ is a stable local basis for $\mathcal{S}$ in the sense that both (1.3) and (1.4) hold.

For a given spline space $\mathcal{S}$, there are generally many different minimal determining sets $\mathcal{M}$. However, designing algorithms which produce stable local minimal determining sets is nontrivial, in general.

4. Type-O$_\theta$ triangulations. Given $\theta > 0$, let $\mathcal{T}_\theta$ be the set of all triangulations whose smallest angle is at least θ. In order to construct stable local bases for the spline spaces $\mathcal{S}_4^1(\triangle)$, we need to work with a restricted subclass of $\mathcal{T}_\theta$. First we introduce some terminology.

DEFINITION 4.1. *The* degree $\deg(v)$ *of a vertex* v *is the number of edges attached to it. A vertex* v *is a called a* good vertex *if it is a boundary vertex, an interior vertex of odd degree, or an interior vertex with* $\deg(v) = 4$. *Any other vertex is a* bad vertex.

In [3], good and bad vertices are called terminating vertices and propagating vertices, respectively. A vertex v is bad if and only if it is an interior vertex of even degree at least 6. In the following we will say that a vertex is odd (even) if $\deg(v)$ is odd (even).

For any three vertices u, v, w, we denote by $\angle(u, v, w)$ the smallest of the two angles made by the edges $\langle v, u\rangle$ and $\langle v, w\rangle$ meeting at v.

DEFINITION 4.2. *Suppose* $e := \langle v, z\rangle$ *is the edge between two triangles* $\langle u, v, z\rangle$ *and* $\langle v, w, z\rangle$. *Then* e *is called* θ-near-degenerate at v *provided* $\pi - \angle(u, v, w) < \theta^2/4\pi$. *If* v *is a bad vertex,* z *is a good vertex, and* e *is not* θ*-near-degenerate at* v*, we say that* v *is* θ-supported *(by* z*). We call* e *a* θ-supporting edge *for* v.

Near-degenerate edges were first introduced in [15], and also play a role in [12]. Our definition here is slightly different. The case where $\angle(u, v, w) = \pi$ corresponds to a classical degenerate edge. We are now ready to introduce the class of triangulations of interest in this paper.

DEFINITION 4.3. *We call* $\triangle \in \mathcal{T}_\theta$ *a* type-O_θ *triangulation provided that every bad vertex* v *in* $\triangle$ *is* θ*-supported.*

Type-O_θ triangulations form a subclass of the type-O triangulations introduced in [6,7]. The key difference is that for a type-O triangulation, it is only required that supporting edges be non-degenerate. Not all triangulations are of type-O_θ, of course, but in Sects. 7 and 11 below we present variants of the methods in [6,7], which can be used to convert any given triangulation $\triangle \in \mathcal{T}_\theta$ into a triangulation of type-$O_{\theta/2}$ or type-$O_{\kappa\theta}$ respectively, where κ is an appropriate constant depending only on θ.

5. Minimal determining sets on disks $D_2(v)$. Throughout this section we assume that $\triangle$ is a type-O_θ-triangulation. Following [3], as a first step towards building a stable local basis for $\mathcal{S}_4^1(\triangle)$, in this section we focus on the disks $D_2(v)$ surrounding vertices v of $\triangle$. Given such a disk, we say that $\mathcal{M}_v \subseteq D_2(v)$ is a MDS for $\mathcal{S}_4^1(\triangle)$ on $D_2(v)$ if setting the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_v}$ to arbitrary real numbers, the coefficients $\{c_\xi\}_{\xi\in D_2(v)\setminus\mathcal{M}_v}$ can be uniquely computed by using those smoothness conditions (2.2) that involve only coefficients c_ξ with $\xi \in D_2(v)$. We call such a MDS stable provided that

$$\max_{\xi\in D_2(v)} |c_\xi| \le K \max_{\xi\in\mathcal{M}_v} |c_\xi|,$$

where K is a constant depending only on θ. Let

$$\mathcal{E}_v := \{\xi \in R_2(v) \cap e : \ e \text{ is an edge attached to } v\}. \tag{5.1}$$

Throughout the section we suppose that the vertices attached to v are labelled as v_1, …, v_n in counterclockwise order. If v is an interior vertex, we identify $v_{n+1} := v_1$. Let $T_i := \langle v, v_i, v_{i+1}\rangle$. Recall that a vertex is called singular provided that it is formed by the intersection of two straight lines.

LEMMA 5.1. *If* v *is a good vertex of* $\triangle$*, then there exists a set* $\mathcal{A}_v \subseteq D_2(v) \setminus \mathcal{E}_v$ *such that* $\mathcal{M}_v := \mathcal{A}_v \cup \mathcal{E}_v$ *is a stable MDS for* $\mathcal{S}_4^1(\triangle)$ *on* $D_2(v)$.

Proof. We choose the sets $\mathcal{M}_v$ as in Lemmas 2–5 of [3], but with special care to insure stability. We divide the proof into four cases. For convenience, we define a_i to be the B-coefficients associated with the domain points $\xi_{211}^{T_i}$ for $i = 1, \dots, n$, and let c_0 be the coefficient associated with the domain point at v. In addition, we let b_i

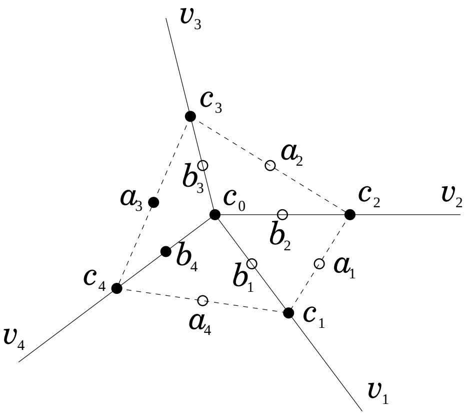

FIG. 1. *B-coefficients in Case 4.*

and c_i be the coefficients associated with the domain points at the intersections of the edges $e_i := \langle v, v_i \rangle$ with the rings $R_1(v)$ and $R_2(v)$, respectively, see Fig. 1.

Case 1: If v is a boundary vertex, we choose

$$\mathcal{A}_v := \{\xi_{400}^{T_1}, \xi_{310}^{T_1}, \xi_{301}^{T_1}, \xi_{211}^{T_1}\}. \tag{5.2}$$

Suppose that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_v}$ of $s \in \mathcal{S}_4^1(\triangle)$ have been set. Then clearly using the smoothness conditions (2.2), we can compute all remaining coefficients in $D_2(v)$. Their computation is stable since the size of the multipliers in the smoothness conditions is controlled by the smallest angle in $\triangle$. Thus $\mathcal{M}_v$ is a stable MDS for $\mathcal{S}_4^1(\triangle)$ on $D_2(v)$.

Case 2: If v is an interior vertex of odd degree, we choose

$$\mathcal{A}_v := \{\xi_{400}^{T_1}, \xi_{310}^{T_1}, \xi_{301}^{T_1}\}. \tag{5.3}$$

Assume that the coefficients $\{c_\xi\}_{\xi \in \mathcal{M}_v}$ of $s \in \mathcal{S}_4^1(\triangle)$ have been set. The smoothness conditions stably determine the remaining coefficients in $D_1(v)$. That leaves n remaining coefficients $a_1, \ldots, a_n$ on $R_2(v)$, where a_i is associated with the domain point $\xi_{211}^{T_i}$. Assuming as in [3]

$$v_j = \alpha_j v_{j-2} + \beta_j v_{j-1} + \gamma_j v, \tag{5.4}$$

then writing down the C^1 smoothness conditions across each of the edges $\langle v, v_i \rangle$ leads to the system

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -\alpha_1 & 1 \\ 1 & 0 & \cdots & 0 & 0 & -\alpha_2 \\ -\alpha_3 & 1 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots & \\ 0 & 0 & \cdots & -\alpha_n & 1 & 0 \end{pmatrix} \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_n \end{pmatrix} = r, \tag{5.5}$$

where the components of the right-hand side r are just combinations of the known coefficients involving the factors β_i and γ_i. As observed in [3], the determinant of this matrix is $1 - \prod_{j=1}^n \alpha_j$. It is known (and easy to see) that since n is odd, $\prod_{j=1}^n \alpha_j = -1$, and thus the determinant is 2. But then applying Cramer's rule, we immediately see that the computation of a_i is stable, *i.e.* $|a_i|$ is bounded by a constant times the size of the set coefficients, where the constant depends only on the smallest angle $\theta_\triangle$ in $\triangle$.

Case 3: If v is a singular vertex, we choose $\mathcal{A}_v$ as in (5.2). Assume that the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_v}$ of $s \in \mathcal{S}_4^1(\triangle)$ have been set. The smoothness conditions (2.2) stably determine the remaining coefficients in $D_1(v)$. Then there are three remaining coefficients a_2, a_3, a_4, where a_i is associated with the domain point $\xi_{211}^{T_i}$. As in Case 1, these coefficients can be stably computed from the smoothness conditions (2.2). Although there are more conditions than unknowns in this case, a_2, a_3, a_4 are *uniquely* determined since as shown in [3], $\mathcal{A}_v \cup \mathcal{E}_v$ is a MDS for $\mathcal{S}_4^1(\triangle)$ on $D_2(v)$.

Case 4: If v is a nonsingular interior vertex with $\deg(v) = 4$, we choose

$$\mathcal{A}_v := \{\xi_{400}^{T_4}, \xi_{310}^{T_4}, \xi_{211}^{T_3}\}, \tag{5.6}$$

where, without loss of generality, we suppose that $\langle v, v_1\rangle$ is a "best edge" in the sense that

$$|\beta_2| = \max_{1\le i\le 4} |\beta_i|, \tag{5.7}$$

and that

$$\gamma_3 \ge 1. \tag{5.8}$$

To justify this last inequality, we note that the definition of barycentric coordinates as ratios of areas implies $\gamma_1 A_3 + \gamma_3 A_1 = A_1 + A_2 + A_3 + A_4$, where A_i is the area of the triangle T_i, $i = 1, \dots, 4$. Hence,

$$\max\{\gamma_1, \gamma_3\} \ge \frac{\gamma_1 A_3 + \gamma_3 A_1}{A_3 + A_1} > 1.$$

Labelling the B-coefficients as in Fig. 1, assume the coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_v} = \{a_3, b_4, c_0, c_1, c_2, c_3, c_4\}$ of $s \in \mathcal{S}_4^1(\triangle)$ have been set. (They are marked with black dots in the figure.) Then as shown in Lemma 5 of [3], the remaining coefficients associated with domain points in $D_2(v)$ are uniquely determined. We now show that they can be *stably* computed.

First, we compute a_4 from the C^1 smoothness condition

$$a_4 = \alpha_1 a_3 + \beta_1 c_4 + \gamma_1 b_4, \tag{5.9}$$

across the edge e_4. This computation is stable since $\alpha_1, \beta_1, \gamma_1$ are bounded by a constant depending only on θ. Then writing down five of the seven C^1 smoothness conditions involving the unknown coefficients leads to the system

$$\begin{pmatrix} \gamma_2 & -1 & 0 & 0 & 0 \\ \beta_2 & 0 & -1 & 0 & 0 \\ 0 & \alpha_3 & \gamma_3 & -1 & 0 \\ \alpha_3 & 0 & \beta_3 & 0 & -1 \\ 0 & 0 & 0 & \alpha_4 & \gamma_4 \end{pmatrix} \begin{pmatrix} b_1 \\ a_1 \\ b_2 \\ a_2 \\ b_3 \end{pmatrix} = \begin{pmatrix} -\alpha_2 a_4 - \beta_2 c_1 \\ -\alpha_2 b_4 - \gamma_2 c_0 \\ -\beta_3 c_2 \\ -\gamma_3 c_0 \\ a_3 - \beta_4 c_3 \end{pmatrix}.$$

As shown in [3], the determinant of this system is given by $D = 2\alpha_4\beta_2\gamma_3$. Since $\alpha_4 = -A_3/A_2$, $|\alpha_4^{-1}|$ is bounded above by a constant depending only on the smallest angle in the triangulation, see Lemma 3.2 of [15]. Taking into account (5.8), we obtain

$$|D^{-1}| \le K_1|\beta_2|^{-1}, \tag{5.10}$$

where K_1 depends only on the smallest angle in the triangulation. (Recall that $\beta_i = 0$ if and only if e_{i-1} is degenerate at v. Since v is nonsingular, we have $\beta_2 \neq 0$.)

Using Cramer's rule, we obtain

$$\begin{aligned} b_1 = -D^{-1}\Big(&\alpha_3\alpha_4\beta_2 c_1 + \alpha_4\beta_3 c_2 + \beta_4 c_3 + \alpha_2\alpha_3\alpha_4\beta_1 c_4 \\ &+ (\gamma_2\gamma_4\beta_3 + \gamma_3(\gamma_2\alpha_4 + \gamma_4))c_0 + (\alpha_2\gamma_4\beta_3 + \alpha_2\alpha_4(\gamma_1\alpha_3 + \gamma_3))\, b_4 \\ &+ (\alpha_1\alpha_2\alpha_3\alpha_4 - 1)\, a_3\Big). \end{aligned}$$

Since $\alpha_i = -A_{i-1}/A_{i-2}$, $i = 1, \ldots, 4$, we have $\alpha_1\alpha_2\alpha_3\alpha_4 = 1$. By using identities (26) of [3], it is easy to show that

$$\gamma_2\alpha_4 + \gamma_4 = \alpha_2^{-1}\alpha_3^{-1}\beta_2(\gamma_3 + \beta_3\gamma_2), \qquad \gamma_1\alpha_3 + \gamma_3 = \alpha_1^{-1}\alpha_2^{-1}\beta_1(\gamma_2 + \beta_2\gamma_1).$$

This, together with (5.7) and (5.10) implies

$$|b_1| \le K_2 \max\{c_\xi :\ \xi \in \mathcal{M}_v\},$$

where K_2 depends only on the smallest angle θ. At this point we have computed a_4 and b_1 stably. But then we can stably compute the coefficients b_2, b_3 and then a_1, a_2 directly from the C^1 smoothness conditions (2.2). □

EXAMPLE 5.2. *In Case 4 of the above lemma it is essential to choose a "best edge" satisfying (5.7).*

Discussion. Let v be a nonsingular interior vertex with $\deg(v) = 4$, and let $\mathcal{A}_v$ be the set in (5.6). According to [3], $\mathcal{M}_v = \mathcal{A}_v \cup \mathcal{E}_v$ is a MDS for $\mathcal{S}_4^1(\triangle)$ on $D_2(v)$ as soon as $\beta_2 \neq 0$ and $\gamma_3 \neq 0$. Suppose that all coefficients $\{c_\xi\}_{\xi\in\mathcal{M}_v}$ of $s \in \mathcal{S}_4^1(\triangle)$ are zero except $c_4 = 1$. Then the calculation in Case 4 shows that

$$|b_1| = \Big| -\frac{\alpha_2\alpha_3\beta_1}{2\beta_2\gamma_3}\Big| \ge K_3\Big|\frac{\beta_1}{\beta_2}\Big| \ge K_4\Big|\frac{\sin(\theta_3+\theta_4)}{\sin(\theta_4+\theta_1)}\Big|,$$

where K_3, K_4 depend only on θ, and $\theta_i := \angle(v_i, v, v_{i+1})$, $i = 1, \ldots, 4$. Hence, if $\mathcal{A}_v$ were chosen without inforcing (5.7), then $|b_1|$ could be arbitrarily large, depending on the exact geometry of the triangles attached to v. □

Lemma 5.1 does not hold in general for bad vertices (see Example 1 of [3]). In this case we have to be satisfied with a MDS which contains all but one of the points in (5.1).

LEMMA 5.3. *Suppose v is a bad vertex of $\triangle$ that is θ-supported by the vertex v_3. Let $\mathcal{A}_v$ be the set of domain points in (5.2). Then*

$$\mathcal{M}_v := \mathcal{A}_v \cup \big(\mathcal{E}_v \setminus \{e_3 \cap R_2(v)\}\big)$$

is a stable MDS for $\mathcal{S}_4^1(\triangle)$ on $D_2(v)$.

Proof. Assume that the coefficients of $s \in \mathcal{S}_4^1(\triangle)$ have been set. Suppose we number the coefficients in $D_2(v)$ as in the proof of Lemma 5.1. Then the smoothness conditions stably determine the remaining coefficients in $D_1(v)$ and all of the coefficients on $R_2(v)$ except for the coefficient c_3. Then with $\alpha_4, \beta_4, \gamma_4$ as in (5.4), we find that

$$c_3 = \frac{a_3 - \alpha_4 a_2 - \gamma_4 b_3}{\beta_4}.$$

Since e_3 is a θ-supporting edge for v, it follows that $|\beta_4|$ is bounded away from 0 by a constant depending only on θ. □

It is easy to see that the $\mathcal{M}_v$ of Lemma 5.3 is not guaranteed to be a stable MDS if we only assume that e_3 is non-degenerate at v. Indeed, in that case $|\beta_4|$ may be arbitrarily small, thus making $|c_3|$ arbitrarily large.

6. Stable Local Bases for $\mathcal{S}_4^1(\triangle)$ on type-O$_\theta$ triangulations.

THEOREM 6.1. *Suppose $\triangle$ is a type-O$_\theta$ triangulation, and let $\mathcal{M}$ be the union of the following sets of domain points:*

1) *for each vertex v of $\triangle$, include the set $\mathcal{A}_v$ described in Lemmas 5.1 and 5.3, depending on whether v is a good or a bad vertex,*
2) *the point $(v+u)/2$ for each edge $\langle v, u\rangle$ of $\triangle$, except for one θ-supporting edge e_v for every bad vertex v.*

Then $\mathcal{M}$ is a stable local minimal determining set for $\mathcal{S}_4^1(\triangle)$.

Proof. First we show that $\mathcal{M}$ is a determining set. Suppose that $s \in \mathcal{S}_4^1(\triangle)$ and that we set all of its coefficients corresponding to domain points in $\mathcal{M}$ to zero. Then by Lemma 5.3 it follows that all coefficients of s must be zero for domain points in the disks $D_2(v)$ where v is a bad vertex. But then by Lemma 5.1, all coefficients of s must also be zero for domain points in the disks $D_2(v)$ where v is a good vertex. This shows that all coefficients of s must be zero, and we conclude that $\mathcal{M}$ is a determining set.

To see that $\mathcal{M}$ is minimal, we compare its cardinality with the known dimension of $\mathcal{S}_4^1(\triangle)$. By [3],

$$\dim \mathcal{S}_4^1(\triangle) = 3V_I + 4V_B + E + \sigma, \tag{6.1}$$

where V_I, V_B are the number of interior and boundary vertices of $\triangle$, E is the number of edges, and σ is the number of singular vertices. Now each of the sets $\mathcal{A}_v$ contains 4 points whenever v is a boundary vertex or a singular interior vertex. It contains 3 points for all other good vertices, and 4 points for all bad vertices. Since $\mathcal{M}$ includes the center of each edge except for one supporting edge attached to each bad vertex, we see that the cardinality of $\mathcal{M}$ is precisely the number in (6.1). This implies that $\mathcal{M}$ is minimal.

Let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the corresponding dual basis splines satisfying (3.1). In view of the nature of $\mathcal{M}$, it is easy to see that the B_ξ are locally supported. In particular,

1) if ξ is a point in one of the sets $\mathcal{A}_v$ as in item 1) of the theorem and v is a good vertex, then B_ξ has support on $\mathrm{star}(v)$,
2) if ξ is a point in one of the sets $\mathcal{A}_v$ as in item 1) of the theorem and v is a bad vertex that is θ-supported by u, then B_ξ has support on $\mathrm{star}(v) \cup \mathrm{star}(u)$,
3) if ξ is a point at the midpoint of an edge $\langle v, u\rangle$ as in item 2) of the theorem, then B_ξ has support on $\mathrm{star}(v) \cup \mathrm{star}(u)$.

This shows that (1.3) holds with $\ell = 2$, and completes the proof of locality. Concerning stability, we note that (3.2) follows immediately from the proofs of Lemmas 5.1 and 5.3. □

7. Creating type-$\mathrm{O}_{\theta/2}$ triangulations with Clough-Tocher refinement. In this section we show that the following algorithm from [6] can be used to convert an arbitrary triangulation $\triangle$ in $\mathcal{T}_\theta$ into a type-$\mathrm{O}_{\theta/2}$ triangulation.

ALGORITHM 7.1. *Suppose $\triangle \in \mathcal{T}_\theta$. and let $\triangle^{(0)} := \triangle$.*

```
Do for i = 0,...,
    Stop if the set U_i of bad vertices of △^(i) that are
      not θ-supported is empty
    Choose a vertex v ∈ U_i
    Choose a triangle T attached to v
    Split T into three triangles about its centroid, and
      let △^(i+1) be the resulting triangulation
```

THEOREM 7.2. *Algorithm 7.1 terminates after a finite number of steps, and the final triangulation is a type-$\mathrm{O}_{\theta/2}$ triangulation.*

Proof. First we consider a typical step where $T := \langle u, v, w\rangle$ is the triangle being split about its centroid z, and v is the bad vertex that is not θ-supported. Then after splitting T, clearly the smallest angle is at least $\theta/2$. Moreover, we claim that each of the vertices u, v, w, z is either good, or is a θ-supported bad vertex. Indeed, $\deg(z) = 3$ and v changes from even to odd, and so both z and v are now certainly good. Now u may have switched from odd to even, so that it is now a bad vertex, but since $\angle(w, u, v) \le \pi - 2\theta$, it follows that $\pi - \angle(w, u, v) \ge 2\theta > \theta^2/4\pi$, and thus u is θ-supported by z. A similar argument shows that w is either good, or is θ-supported by z.

The fact that the algorithm terminates after a finite number of steps follows from the observation that at least one nonsupported bad vertex is eliminated in each step. Clearly, if a triangle is split, then none of its subtriangles will have to be split in a later step. This guarantees that the smallest angle of the final triangulation $\triangle^{(M)}$ is at least $\theta/2$. Since every θ-supported vertex is automatically $\theta/2$-supported, we conclude that $\triangle^{(M)}$ is of type-$\mathrm{O}_{\theta/2}$. □

8. Edge swapping. It was shown in [7] that an arbitrary triangulation can be converted to a type-O triangulation by performing a finite number of edge swaps. However, as noted above, the class of type-O triangulations is too large to insure the existence of stable local bases, and thus in Sect. 11 below we present a related algorithm for converting triangulations $\triangle \in \mathcal{T}_\theta$ into type-$\mathrm{O}_{\kappa\theta}$ triangulations with an appropriate κ. In this section we collect several preliminary results about edge swapping.

Given a triangulation $\triangle$, suppose $T_1 := \langle u_1, u_2, u_4\rangle$ and $T_2 := \langle u_3, u_4, u_2\rangle$ are two triangles in $\triangle$ sharing the interior edge $e := \langle u_2, u_4\rangle$. The edge e is called swappable if replacing the edge e by the edge $\tilde{e} := \langle u_1, u_3\rangle$ leads to a nontrivial triangulation $\widetilde{T}_1 := \langle u_1, u_2, u_3\rangle$ and $\widetilde{T}_2 := \langle u_1, u_3, u_4\rangle$. Obviously, e is swappable if and only if the quadrilateral $Q := T_1 \cup T_2$ is convex and e is not degenerate at either end. Swapping edges in a triangulation $\triangle$ leads to a new triangulation $\widetilde{\triangle}$ with the same vertices.

EXAMPLE 8.1. *Swapping can lead to the introduction of small angles.*

Discussion. Let T_1, T_2 be a pair of triangles as shown on the left in Fig. 2. The smallest angle in this triangulation is $\pi/4$. The result of replacing the edge

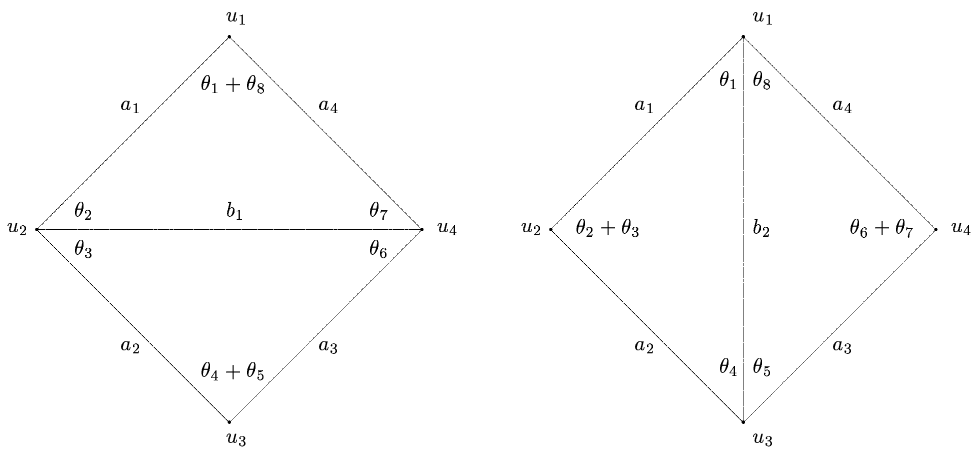

FIG. 2. *Swapping an edge of a triangulation.*

$e := \langle u_2, u_4\rangle$ by $\tilde{e} := \langle u_1, u_3\rangle$ is shown on the right. Its smallest angle is also $\pi/4$. Now consider the same configuration with u_2 closer to the edge $\tilde{e}$. The smallest angle in the triangulation on the left is still $\pi/4$, but the angles θ_1, θ_4 and thus the smallest angle in the triangulation on the right can be arbitrarily small if u_2 is close enough to $\tilde{e}$. □

It is clear from this example that to prevent the creation of small angles, we have to restrict the swapping process to quadrilaterals where the angles $\theta_2+\theta_3$ and $\theta_6+\theta_7$ in Fig. 2 are bounded away from π.

LEMMA 8.2. *Let Q be a quadrilateral as shown on the left in Fig. 2. Suppose that the smallest angle in Q is θ, and that $\theta_2+\theta_3 \le \pi - \kappa_1\theta$ and $\theta_6+\theta_7 \le \pi-\kappa_2\theta$, where $\kappa_1, \kappa_2 > 0$. Then after swapping the interior edge, the smallest angle in the resulting triangulation is at least $\kappa\theta$ where $\kappa = \min\{\kappa_1,\kappa_2\}/(1+(\pi/2)\sin^{-2}\theta)$.*

Proof. Note that $\theta \le \pi/3$ as $\pi/3$ is the largest smallest angle any triangle can have. Let a_i and b_i be the side lengths indicated in Fig. 2. Applying the *law of sines*, we have

$$\frac{\sin\theta_4}{a_1} = \frac{\sin\theta_1}{a_2}, \qquad \frac{\sin(\theta_1+\theta_8)}{b_1} = \frac{\sin\theta_7}{a_1}, \qquad \frac{\sin(\theta_4+\theta_5)}{b_1} = \frac{\sin\theta_6}{a_2}. \tag{8.1}$$

This implies

$$\frac{\sin\theta_1}{\sin\theta_4} = \frac{\sin\theta_6}{\sin\theta_7}\,\frac{\sin(\theta_1+\theta_8)}{\sin(\theta_4+\theta_5)}.$$

It follows that

$$\sin^2\theta \le \frac{\sin\theta_1}{\sin\theta_4} \le \frac{1}{\sin^2\theta}.$$

Using the inequality $2x/\pi \le \sin x \le x$, we get

$$\frac{2\sin^2\theta}{\pi} \le \frac{\theta_1}{\theta_4} \le \frac{\pi}{2\sin^2\theta}$$

or equivalently,

$$\theta_1 \le (\pi/2)\theta_4\sin^{-2}\theta, \qquad \theta_4 \le (\pi/2)\theta_1\sin^{-2}\theta.$$

Now the hypothesis implies $\theta_1 + \theta_4 \geq \kappa_1\theta$, and it follows that

$$\theta_1, \theta_4 \geq \kappa_1\theta/(1 + (\pi/2)\sin^{-2}\theta).$$

Using $\theta_5 + \theta_8 \geq \kappa_2\theta$, a similar argument shows that

$$\theta_5, \theta_8 \geq \kappa_2\theta/(1 + (\pi/2)\sin^{-2}\theta),$$

and the result follows. □

9. *n*-Cells. Suppose $\triangle_v$ is a triangulation with exactly one interior vertex v and n boundary vertices. Such a triangulation is commonly called an n-cell. Suppose the boundary vertices are numbered in counterclockwise order as $v_1, \ldots, v_n$, and identify $v_{n+1} = v_1$. For each $i = 1, \ldots, n$, let $e_i := \langle v, v_i \rangle$, $T_i := \langle v, v_i, v_{i+1} \rangle$, $\theta_i := \angle(v_i, v, v_{i+1})$, $\varphi_i := \angle(v, v_i, v_{i+1}) + \angle(v_{i-1}, v_i, v)$, and $\omega_i := \theta_i + \theta_{i-1}$.

LEMMA 9.1. *Suppose $\triangle_v$ is an n-cell with $n \geq 6$ whose smallest angle is at least θ. Then*

1) *at least three of the φ_i satisfy $\varphi_i \leq \pi - 4\theta/(n-2)$,*
2) *at most two of the ω_i satisfy $\omega_i > \pi - \theta$,*
3) *if $\omega_i, \omega_j > \pi - \theta$, then $|i - j| = 1$, i.e., ω_i and ω_j correspond to consecutive edges.*

Proof. To prove 1), suppose to the contrary that $n-2$ of the φ_i are greater than $\pi - 4\theta/(n-2)$. But then

$$(n-2)\pi = \sum_{i=1}^{n} \varphi_i > (n-2)\big(\pi - \frac{4\theta}{n-2}\big) + 4\theta = (n-2)\pi,$$

and this contradiction implies 1).

To prove 2), suppose to the contrary that $\omega_k, \omega_l, \omega_m > \pi - \theta$. Then at least two of these do not overlap, say ω_k, ω_l. That leaves $n-4$ of the angles θ_i that are not covered by ω_k or ω_l, which leads to the contradiction

$$2\pi \geq \omega_k + \omega_l + (n-4)\theta > 2\pi - 2\theta + (n-4)\theta \geq 2\pi.$$

The same argument proves 3) since ω_i, ω_j do not overlap unless they correspond to consecutive edges.□

COROLLARY 9.2. *Suppose $\triangle_v$ is an n-cell with $n \geq 6$ whose smallest angle is at least θ. Then at least one of the edges e_i is not θ-near-degenerate at either end.*

Proof. Lemma 9.1 implies that for some i, both $\varphi_i \leq \pi - 4\theta/(n-2)$ and $\omega_i \leq \pi - \theta$. Now $n\theta \leq 2\pi$ implies $4/(n-2) > 4/n \geq 2\theta/\pi$, and thus

$$\varphi_i \leq \pi - \frac{4\theta}{n-2} \leq \pi - \frac{2\theta^2}{\pi} \leq \pi - \frac{\theta^2}{4\pi}, \qquad \omega_i \leq \pi - \theta \leq \pi - \frac{\theta^2}{4\pi}.$$

We conclude that the edge e_i is not θ-near-degenerate at either end. □

10. Special vertices. In this section we introduce a special kind of vertex of importance in the following section.

DEFINITION 10.1. *Given a vertex v and associated n-cell $\triangle_v$, we say that v is a* θ-special vertex *provided*

1) *the smallest angle in $\triangle_v$ is at least θ,*

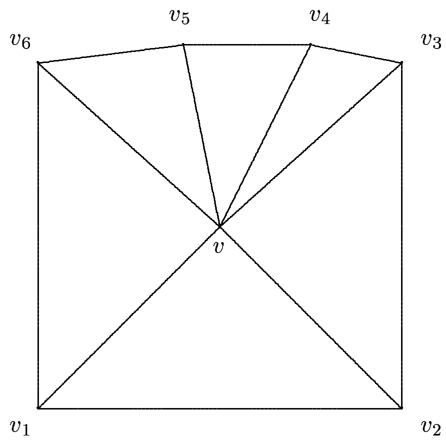

FIG. 3. *A special vertex v.*

2) *$n \geq 6$ is even,*
3) *v_1 and v_2 are odd while $v_3, \ldots, v_n$ are all even,*
4) *e_1 and e_2 are θ-near-degenerate at v,*
5) *e_i is θ-near-degenerate at v_i for $i = 4, \ldots, n-1$.*

Fig. 3 shows an n-cell $\triangle_v$ associated with a special vertex v of degree $n = 6$. If v is a θ-special vertex, then Corollary 9.2 implies that either e_3 or e_n is not θ-near-degenerate at either end. By definition of a θ-special vertex, we observe that $|\pi - \omega_i| < \theta^2/4\pi$ for $i = 1, 2$, and $|\pi - \varphi_i| < \theta^2/4\pi$ for $i = 4, \ldots, n-1$. We also have $\theta_i \geq \theta$ for all i, and recall that $\theta \leq \pi/3$.

LEMMA 10.2. *Suppose v is a θ-special vertex, and let $\widetilde{\triangle}_v$ be the triangulation obtained from $\triangle_v$ after first swapping the edge e_3, and then the edge $\langle v, v_4 \rangle$ of the new quadrilateral $\langle v, v_2, v_4 \rangle \cup \langle v_5, v, v_4 \rangle$. Then the smallest angle in $\widetilde{\triangle}_v$ is at least $\kappa\theta$, where*

$$\kappa := \frac{\theta}{\pi(1 + (\pi/2)\sin^{-2}\ \omega\theta)} \tag{10.1}$$

and

$$\omega := \frac{\theta}{\pi(1 + (\pi/2)\sin^{-2}\theta)}. \tag{10.2}$$

Proof. Corollary 9.2 implies that either e_3 or e_n is not θ-near-degenerate at either end. Without loss of generality, we assume it is edge e_3. In view of Lemma 8.2, after swapping e_3, we get a new triangulation with smallest angle at least $\omega\theta$. Now let $\tilde{\varphi}_4 := \angle(v_5, v_4, v) + \angle(v, v_4, v_2)$ and $\tilde{\omega}_4 := \angle(v_5, v, v_4) + \angle(v_4, v, v_2)$. Since

$$\tilde{\omega}_4 + \omega_1 + \sum_{i=5}^{n-1} \theta_i = 2\pi, \tag{10.3}$$

it follows that

$$\tilde{\omega}_4 < 2\pi - (\pi - \frac{\theta^2}{4\pi}) - \theta = \pi - \theta + \frac{\theta^2}{4\pi} \le \pi - \frac{\theta^2}{4\pi}.$$

Examining the polygon $\langle v, v_2, v_4, v_5, \dots, v_n \rangle$, we see that

$$\tilde{\varphi}_4 + \sum_{i=5}^{n-1} \varphi_i + \angle(v_{n-1}, v_n, v) + (2\pi - \omega_1) + \angle(v, v_2, v_4) = (n-3)\pi. \tag{10.4}$$

Now $\omega_1 < \pi + \theta^2/4\pi$ implies $\omega_1 - 2\pi < -(\pi - \theta^2/4\pi)$. Furthermore, $\varphi_i > \pi - \theta^2/4\pi$, $i = 5, \dots, n-1$, and $\angle(v_{n-1}, v_n, v) \ge \theta$. Substituting this in (10.4), we get

$$\tilde{\varphi}_4 \le (n-3)\pi - (n-4)(\pi - \frac{\theta^2}{4\pi}) - \theta \le \pi + \frac{n\theta^2}{4\pi} - \theta - \frac{4\theta^2}{4\pi} < \pi - \frac{\theta^2}{4\pi}$$

since $n\theta \le 2\pi$. Now Lemma 8.2 implies that after swapping the edge $\langle v, v_4 \rangle$, the smallest angle in the resulting triangulation $\widetilde{\triangle}_v$ is at least $\kappa\theta$ as asserted. □

11. Creating type-$\mathrm{O}_{\kappa\theta}$ triangulations with edge swapping. Swapping the diagonal of a quadrilateral $Q := \langle u, v, w, z \rangle$ in a triangulation changes the degrees of all four vertices of Q. Thus, properly applied, swapping can be used to eliminate bad vertices that are not supported from a triangulation, thus converting it to a type-O triangulation. An appropriate algorithm can be found in [7], but it does not guard against creating triangles with arbitrarily small angles. We now present an algorithm which converts a given triangulation $\triangle \in \mathcal{T}_\theta$ into a type-$\mathrm{O}_{\kappa\theta}$ triangulation, where κ is given in (10.1). For the remainder of the paper we say that an edge of a triangulation $\triangle$ is θ-swappable provided that it is not θ-near-degenerate at either end.

ALGORITHM 11.1. *Given a triangulation $\triangle \in \mathcal{T}_\theta$,*

(I) Do for $i = 0, \dots,$
  Let $\mathcal{U}_i$ be the set of bad vertices v of $\triangle^{(i)}$ such that
    v is not θ-supported and there exists a quadrilateral
    $Q := \langle u, v, w, z \rangle$ formed by two triangles $\langle u, v, z \rangle, \langle v, w, z \rangle$
    in $\triangle^{(i)}$ such that u, v, w, z are all even and
    $e = \langle v, z \rangle$ is θ-swappable
  Stop if the set $\mathcal{U}_i$ is empty; otherwise, swap e and
    let $\triangle^{(i+1)}$ be the resulting triangulation

(II) Let $\triangle^{(N+1)}$ be the final triangulation of the first loop
  Do for $i = N+1, \dots,$
  Let $\mathcal{W}_i$ be the set of θ-special vertices of $\triangle^{(i)}$
  Stop if $\mathcal{W}_i$ is empty; otherwise, choose a vertex $v \in \mathcal{W}_i$,
    perform the double swap of Lemma 10.2, and
    let $\triangle^{(i+1)}$ be the resulting triangulation

THEOREM 11.2. *Algorithm 11.1 terminates after a finite number of steps, and the final triangulation is a type-*$\mathrm{O}_{\kappa\theta}$ *triangulation where κ is given in (10.1).*

Proof. We begin by discussing the first loop. Suppose $v \in \mathcal{U}_i$ and that $Q := \langle u, v, w, z \rangle$ is a quadrilateral with u, v, w, z all even. Then swapping the edge $e := \langle v, z \rangle$ produces a new triangulation $\triangle^{(i+1)}$ in which all of the vertices u, v, w, z are now odd, and thus the vertex v is no longer bad. Clearly, the swap does not introduce any new bad vertices, which also insures that no θ-supported bad vertex of $\triangle^{(i)}$ becomes unsupported. Hence $\#\,\mathcal{U}_{i+1} < \#\,\mathcal{U}_i$. By Lemma 8.2 the smallest angle in $\triangle^{(i+1)}$ is now at least $\omega\theta$, where ω is given in (10.2). We also note that since v is now odd, none of the edges attached to v will be swapped in any later step of the first loop. Since $\mathcal{U}_0$ is finite, it follows that loop I stops after a finite number of steps.

We now claim that if a bad vertex of $\triangle^{(N+1)}$ is not θ-supported, then it must be a θ-special vertex. To see this, consider such a vertex v, and let v_i, e_i, T_i, θ_i, φ_i, and ω_i be as in Sect. 9. Note that:

1) If e_i is θ-swappable, then one of the vertices v_{i-1}, v_i, v_{i+1} must be odd, since otherwise we would have dealt with v in the first loop. The odd vertex cannot be v_i, since otherwise v_i would support v.
2) If v_i is odd, then the edge e_i must be θ-near-degenerate at v, since otherwise v_i supports v.
3) By Corollary 9.2, one of the edges e_i is θ-swappable. Without loss of generality we take it to be e_3. Then by 1), v_3 must be even. It follows that at least one of the vertices v_2 or v_4 must be odd. Without loss of generality, we suppose that v_2 is odd. Then by 2) we know that e_2 is θ-near-degenerate at v. This implies $\omega_2 > \pi - \theta^2/4\pi > \pi - \theta$ and $\omega_2 < \pi + \theta^2/4\pi$.
4) By Lemma 9.1, only one additional ω_i can be greater than $\pi - \theta$, and it can only be ω_1. Thus, $\omega_i \le \pi - \theta$ for $i = 4, \dots, n$. Hence, by 2), $v_4, \dots, v_n$ must be even. But then by 1), $e_4, \dots, e_{n-1}$ cannot be θ-swappable, which implies that each of the $e_4, \dots, e_{n-1}$ is θ-near-degenerate at v_i. In particular $\varphi_i > \pi - \theta^2/4\pi$ for $i = 4, \dots, n-1$.
5) The edge e_n is θ-swappable. We already know that $\omega_n \le \pi - \theta$. Let us show that $\varphi_n \le \pi - \theta^2/4\pi$. By examining the polygon $\langle v_1, v, v_3, v_4, \dots, v_n \rangle$, we see that

$$\sum_{i=4}^{n} \varphi_i + 2\pi - \omega_2 + \angle(v_n, v_1, v) + \angle(v, v_3, v_4) = (n-2)\pi. \tag{11.1}$$

Using $n\theta \le 2\pi$, this implies

$$\varphi_n \le \pi - 2\theta + \frac{(n-3)\theta^2}{4\pi} = \pi - 2\theta + \frac{n\theta^2}{4\pi} - \frac{3\theta^2}{4\pi} < \pi - \frac{3\theta^2}{4\pi} < \pi - \frac{\theta^2}{4\pi}.$$

6) Now by 1) one of the vertices v_{n-1}, v_n, v_1 must be odd, and it can only be v_1. Then by 2) e_1 must be θ-near-degenerate at v.

This completes the proof that all unsupported bad vertices of $\triangle^{(N+1)}$ are θ-special and thus lie in $\mathcal{W}_{N+1}$.

Now suppose we perform the double swap of Lemma 10.2 in an n-cell $\triangle_v$ associated with a $v \in \mathcal{W}_{N+1}$. After the double swap, v becomes an even vertex of degree $n-2$, the vertices v_3 and v_5 change from even to odd, v_2 remains odd, and v_4 remains even. If $n = 6$, this means that v becomes a good vertex. Otherwise, it remains a bad vertex, but since

$$\angle(v_5, v, v_1) \le 2\pi - \omega_2 - 2\theta \le \pi + \frac{\theta^2}{4\pi} - 2\theta \le \pi - \frac{\theta^2}{4\pi}, \tag{11.2}$$

it is now θ-supported by v_2.

We have shown that performing loop II removes the θ-special vertex v. Obviously, no new bad vertices are introduced. We now show that no θ-supported bad vertex becomes unsupported. Only the vertices and interior edges of the polygon $\langle v, v_2, v_3, v_4, v_5\rangle$ are affected by the double swap. Thus, since v_2, v_3, v_5 are now odd, a new unsupported bad vertex could be introduced only if v_4 previously supported a bad vertex, but no longer does. (Since v_4 remains even, this could only happen if v_4 were of degree four.) But in this case v_4 could have supported the vertex v since $\omega_4 \le \pi - \theta$, which contradicts our assumption that v was an unsupported bad vertex.

Since $\mathcal{W}_{N+1}$ is finite, and the cardinality of $\mathcal{W}_i$ is reduced in each pass through the loop, we conclude that loop II terminates after a finite number of steps. Clearly, the final triangulation $\triangle^{(M)}$ contains only good vertices or θ-supported bad vertices, which are automatically $\kappa\theta$-supported since $\kappa < 1$. Lemma 10.2 implies that the smallest angle in $\triangle^{(M)}$ is at least $\kappa\theta$, and we conclude that it is a type-$\mathrm{O}_{\kappa\theta}$ triangulation. □

12. Quasi-Interpolation. In this section we construct a quasi-interpolation operator Q mapping functions defined on Ω to the space of splines $\mathcal{S}_4^1(\triangle)$ on a type-O_θ triangulation of Ω. The operator will produce optimal order approximation of smooth functions.

Suppose $\mathcal{M}$ is the MDS for the space $\mathcal{S}_4^1(\triangle)$ described in Theorem 6.1, and let $\{B_\xi\}_{\xi\in\mathcal{M}}$ be the corresponding dual basis splines satisfying (3.1). Then as shown at the end of Sect. 6, the B_ξ satisfy (1.3) with $\ell = 2$. Let λ_ξ be the linear functionals defined at the beginning of Sect. 3. Then we define

$$Qf = Q_\infty f := \sum_{\xi\in\mathcal{M}} \lambda_\xi(\phi_{f,T_\xi})\, B_\xi, \tag{12.1}$$

where T_ξ is the triangle in which the domain point ξ lies, and where for a general triangle T, $\phi_{f,T}$ is the polynomial of degree 4 that interpolates f at the domain points ξ^T_{ijk}. Similarly, we define

$$Q_p f := \sum_{\xi\in\mathcal{M}} \lambda_\xi(\bar\phi_{f,T_\xi})\, B_\xi, \qquad 1 \le p < \infty, \tag{12.2}$$

where for a general triangle T, $\bar\phi_{f,T}$ is the averaged Taylor polynomial associated with f and T, see [15].

By the duality of the basis,

$$\lambda_\xi(Qf) = \lambda_\xi(\phi_{f,T_\xi}), \qquad \xi \in \mathcal{M}. \tag{12.3}$$

and

$$\lambda_\xi(Q_p f) = \lambda_\xi(\bar\phi_{f,T_\xi}), \qquad \xi \in \mathcal{M}, \tag{12.4}$$

and thus $Qf = f$ and $Q_p f = f$ whenever f is a polynomial of degree 4.

THEOREM 12.1. *Let $1 \le p \le \infty$. Then there exists a constant K depending only on the smallest angle in $\triangle$ such that for all f in the Sobolev space $W_p^{k+1}(\Omega)$ with $0 \le k \le d$,*

$$\|D_x^i D_y^j (f - Q_p f)\|_p \le K |\triangle|^{k+1-i-j} |f|_{k+1,p} \tag{12.5}$$

for $0 \le i + j \le k$. If Ω is nonconvex, K also depends on the Lipschitz constant associated with the boundary of Ω.

Proof. See the proof of Theorem 1.1 in [15]. □

13. Interpolation. The quasi-interpolator Q constructed in (12.1) for C^1 quartic splines on type-O_θ triangulations can be used to solve certain Lagrange and Hermite interpolation problems. Suppose $\triangle$ is a triangulation with vertices $\{v_i\}_{i=1}^n$.

THEOREM 13.1. *Let Q be the operator defined in (12.1). Then for any f defined on Ω,*

$$Qf(v_i) = f(v_i), \qquad i = 1, \dots, n. \tag{13.1}$$

Proof. First we note that for each vertex v of $\triangle$, $\mathcal{M}$ contains the domain point at v. But it is well-known that if ξ is a domain point lying at a vertex v of a triangle, then $\lambda_\xi f = f(v)$. Then (13.1) follows immediately from (12.3). □

To get a result for Hermite interpolation, we replace the Lagrange interpolation polynomial $\phi_{f,T}$ by a Hermite interpolation polynomial. Given a triangle T in $\triangle$, let $\tilde{\phi}_{f,T}$ be the polynomial of degree 4 that interpolates gradient information at each of the three vertices of T along with point values at the domain points $\{\xi_{ijk}^T\}$ that do not lie in the disks of radius one around the vertices.

THEOREM 13.2. *Let $\tilde{Q}$ be the operator defined as in (12.1) based on $\tilde{\phi}_{f,T_\xi}$. Then for any f defined on Ω, $\tilde{Q}f$ interpolates f at the vertices of $\triangle$ as in (13.1), and*

$$D_x^i D_y^j \tilde{Q} f(v) = D_x^i D_y^j f(v), \qquad i + j = 1, \tag{13.2}$$

for all vertices v of $\triangle$ except for those that are nonsingular of degree four.

Proof. By construction, the minimal determining set $\mathcal{M}$ contains all vertices v of $\triangle$, and also two additional points on the ring $R_1(v)$ for all $v \in \triangle$ except for those vertices that are nonsingular and of degree 4. Thus, if v is not such an exceptional vertex, then the gradient of $\tilde{Q}f$ at v matches the gradient of $\tilde{\phi}_{f,T}$ at v, which in turn matches the gradient of f there. □

14. Remarks.

REMARK 14.1. Although the local bases in [6,7] are stable for certain triangulations (*e.g.* the three-directional mesh modified by either Clough-Tocher splits or swapping), they are not stable in general. Consider, for example, the type-O triangulations in Fig. 4 obtained from a slightly deformed three-directional mesh. The edge e is not a supporting edge for the bad vertex v since it is ω-near degenerate at v for a very small ω. If $\omega \to 0$, then some of the basis functions corresponding to the points in $D_2(v)$ are unbounded, which shows that the basis is unstable. In addition, if a type-O triangulation includes a near-singular vertex w, then basis functions corresponding to $D_2(w)$ may also be unstable unless $\mathcal{A}_w$ are chosen in accordance with the procedure described in Case 4 of Lemma 5.1 (cf. Example 5.2).

REMARK 14.2. For $d \geq 3r + 2$, the spaces $\mathcal{S}_d^r(\triangle)$ are well-behaved for arbitrary triangulations, and there are well-known results [8,12,15] on stable local bases along with associated interpolants and quasi-interpolants which deliver full approximation power. See also [11], where a Hermite interpolation operator with generally non-stable minimally supported fundamental functions is constructed and shown to possess full approximation power. If $d \geq 4r + 1$, then well-known finite-element interpolation operators have full approximation power, see *e.g.* [9].

REMARK 14.3. As observed in [7], in converting an arbitrary triangulation with smallest angle θ to a type-O triangulation, swapping has a slight advantage over inserting Clough-Tocher splits in that swapping does not introduce any new vertices (which

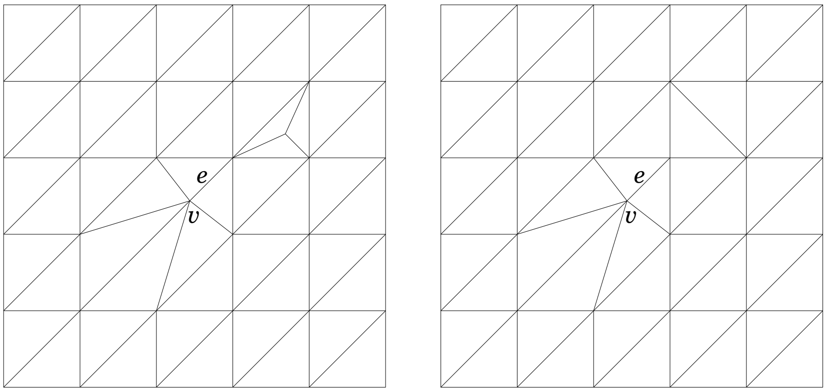

FIG. 4. *Type-O triangulations obtained by the algorithms of [6,7].*

for interpolation would require extra data values which might have to be estimated). The same observation applies to our algorithms, of course.

REMARK 14.4. It is possible to solve the Hermite interpolation problem (13.2) for *all* vertices of a type-O_θ triangulation if "type-O_θ" is understood in a slightly different manner, namely, that all non-singular vertices of degree 4 are also treated as "bad vertices", see [6]. Indeed, then the operator Q of Theorem 13.2 satisfies (13.2) for all vertices. Moreover, any given triangulation with minimal angle θ can be transformed into a type-$O_{\theta/2}$ triangulation of this kind by using Clough-Tocher refinement as in Section 5.

REMARK 14.5. The methods described here can be adapted to perform interpolation of scattered data on the sphere using the class of spherical splines introduced and studied in [1,2]. We leave the details for a future paper.

REMARK 14.6. A *pure* Lagrange interpolation scheme for C^1 quartic bivariate splines has been recently constructed in [10]. For other work on interpolation and approximation with C^1 quartic splines on nonuniform triangulations, see [13,16,17].

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