Interpolation by C^1 Splines of Degree $q \geq 4$ on Triangulations

Oleg Davydov* and Günther Nürnberger

Abstract

Let Δ be an arbitrary regular triangulation of a simply connected compact polygonal domain $\Omega \subset \mathbb{R}^2$ and let $S_q^1(\Delta)$ denote the space of bivariate polynomial splines of degree q and smoothness 1 with respect to Δ. We develop an algorithm for constructing point sets admissible for Lagrange interpolation by $S_q^1(\Delta)$ if $q \geq 4$. In the case $q = 4$ it may be necessary to slightly modify Δ, but only if exceptional constellations of triangles occur. Hermite interpolation schemes are obtained as limits of the Lagrange interpolation sets.

AMS classification: 41A15; 41A63

Keywords: Bivariate splines; Interpolation

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected compact domain with polygonal Jordan boundary, and let Δ denote a regular triangulation of Ω, *i.e.*, Δ is a set of (closed) triangles whose union coincides with Ω such that the intersection of any two triangles in Δ is either empty set, or a common vertex, or a common edge. The space of *bivariate splines* of degree q and smoothness r with respect to Δ is defined by

$$S_q^r(\Delta) := \{s \in C^r(\Omega) \,:\, s_{|T} \in \Pi_q \quad \text{for all} \quad T \in \Delta\}\,, \quad 0 \leq r < q\,,$$

where

$$\Pi_q := \operatorname{span}\{x^i y^j \,:\, i \geq 0,\ j \geq 0,\ i + j \leq q\}$$

is the space of *bivariate polynomials* of total degree q.

In the literature, point sets that admit unique Lagrange and Hermite interpolation by spaces $S_q^r(\Delta^c)$ were constructed for crosscut partitions Δ^c, in particular for Δ^1 and Δ^2-triangulations [1, 4, 14, 19, 20, 21, 25, 26]. Results on the approximation order of these interpolation methods were given in [4, 9, 14, 18, 19, 22, 25, 26].

*Supported in part by a Research Fellowship from the Alexander von Humboldt Foundation.

Much less has been known about interpolation by $S_q^r(\Delta)$, $r \geq 1$, if Δ is an arbitrary triangulation. (Recall that even the dimension of the space $S_q^r(\Delta)$ is known only if $r = 1$, $q \geq 4$ or $r \geq 2$, $q \geq 3r + 2$.) An explicit Hermite interpolation scheme for $S_q^1(\Delta)$, $q \geq 5$, can be obtained by using the nodal basis of this space constructed in [15]. However, this sheds no light on the problem of constructing Lagrange interpolation sets. Recently it was shown [23, 24] that a natural multivariate analogue of well known Schoenberg-Whitney theorem holds true for *almost interpolation sets*, *i.e.*, point sets that can be transformed into a Lagrange interpolation set by an arbitrary small perturbation. Several Schoenberg-Whitney type characterizations of almost interpolation can be found in [7, 11, 12, 13, 23, 24]. A particularly simple characterization theorem is available [13] for spline spaces that admit a locally linearly independent basis. Since the spaces $S_q^1(\Delta)$, $q \geq 5$, are of this type (see [8]), general algorithms of transforming almost interpolation sets into interpolation sets [7, 11, 23] can be applied to construct Lagrange interpolation sets for these spaces. For $q = 4$ it was shown in [2] that a spline in $S_4^1(\Delta)$ exists which coincides with a given function at the vertices of Δ. Under certain restrictions on the triangulation, analogous results were obtained in [5, 6, 16] for function and gradient values at the vertices. (Note that the dimension of $S_4^1(\Delta)$ is about six times the number of vertices of Δ.)

Thus, no explicit Lagrange interpolation schemes for $S_q^1(\Delta)$ and arbitrary triangulations Δ were given in the literature. In this paper we describe an algorithm for constructing such interpolation schemes for $q \geq 4$. In the case $q = 4$ it may be necessary to slightly modify Δ, but only if exceptional local constellations of triangles occur. In these cases we simply split one of the triangles or perturb one of the vertices. Our algorithm for constructing interpolation points is inductive. Starting with one triangle, in each step we add one vertex to the subtriangulation considered before, and simultaneously choose interpolation points on the newly added triangles. Therefore, the interpolating spline can also be computed step by step, by solving small systems of linear equations. By shifting interpolation points to the vertices, we also obtain Hermite interpolation schemes. It is important to note that for Δ^1 and Δ^2 triangulations our method leads to the interpolation schemes with (nearly) optimal approximation order developed in [18] and [22], respectively. In addition, our recent numerical results confirm that our interpolation methods yield nearly optimal approximation order for $S_q^1(\Delta)$, $q \geq 4$, and general classes of triangulations (see [10]).

The paper is organized as follows. Section 2 contains some auxiliary concepts and results. In Section 3 and Section 4 we describe our interpolation schemes for $S_4^1(\Delta)$ and $S_q^1(\Delta)$, $q \geq 5$, respectively. Finally, in Section 5 we give the proofs of the main results.

2 Preliminaries

To simplify notation, we set

$$d_q := \dim \Pi_q = \frac{(q+1)(q+2)}{2}\,, \quad q = 0, 1, \ldots\,. \tag{2.1}$$

Given a regular triangulation Δ, we denote by N the number of triangles, by V_I and V_B the number of interior and boundary vertices respectively, and by E_I and E_B the number of interior and boundary edges respectively. It is well known that

$$\begin{aligned} E_B &= V_B\,, \\ E_I &= 3V_I + V_B - 3\,, \\ N &= 2V_I + V_B - 2\,. \end{aligned} \tag{2.2}$$

It was shown by Morgan & Scott [15] (for $q \geq 5$) and by Alfeld, Piper & Schumaker [2] (for $q = 4$) that

$$\dim S_q^1(\Delta) = d_q + d_{q-2}E_I - (d_q - d_1)V_I + \sigma\,, \quad q \geq 4\,, \tag{2.3}$$

where σ is the number of *singular vertices* of Δ, *i.e.*, those interior vertices for which the adjacent edges of each attached edge are collinear, so that exactly four triangles share a singular vertex and their union is a quadrilateral with the diagonals drawn in (see Fig. 2.1).

A finite set of points $\{z_1, \ldots, z_n\} \subset \Omega$, where $n = \dim S_q^1(\Delta)$, is said to be a *Lagrange interpolation set* for $S_q^1(\Delta)$ if it is admissible for Lagrange interpolation from $S_q^1(\Delta)$, *i.e.*, for any continuous real function f on Ω there exists a unique spline $s \in S_q^1(\Delta)$ satisfying the following *Lagrange interpolation conditions*

$$s(z_i) = f(z_i)\,, \quad i = 1, \ldots, n\,. \tag{2.4}$$

If we consider not only function values of f but also partial derivatives, then we speak of *Hermite interpolation conditions.*

Any finite set $\Lambda \subset \Omega$ is said to be *total* with respect to a subspace $S \subset S_q^1(\Delta)$ if the only spline $s \in S$ vanishing at all points $z \in \Lambda$, is the zero function. It follows easily from basic linear algebra, that Λ is a Lagrange interpolation set for $S_q^1(\Delta)$ if and only if it is total w.r.t. $S_q^1(\Delta)$ and the number of points in Λ is equal to the dimension of $S_q^1(\Delta)$.

Our construction of interpolation sets depends on the following local properties of the triangulation.

Definition 2.1 [2] Suppose e', e, e'' are three consecutive edges attached to a vertex v. The edge e is said to be *degenerate* (at v) whenever the edges e' and e'' are collinear. Otherwise e is *nondegenerate* (at v). (See Fig. 2.2.)

Definition 2.2 The union of all triangles sharing one common vertex v is called the *star* of v, denoted by $ST(v)$.

For any subtriangulation $\Delta' \subset \Delta$ we set

$$\Omega_{\Delta'} := \bigcup_{T \in \Delta'} T\,.$$

Definition 2.3 Let $\Delta' \subset \Delta$. We say that a vertex $v \in \operatorname{int}\Omega \cap \operatorname{bd}\Omega_{\Delta'}$ is *semisingular of type I* with respect to Δ' if $ST(v) \setminus \Omega_{\Delta'}$ includes precisely two triangles $T_1, T_2 \in \Delta \setminus \Delta'$ with a common edge e which is degenerate at v (see Fig. 2.3, a)). A vertex $v \in \operatorname{int}\Omega \cap \operatorname{bd}\Omega_{\Delta'}$ is said to be *semisingular of type II* with respect to Δ' if $ST(v) \setminus \Omega_{\Delta'}$ includes precisely three consecutive triangles $T_1, T_2, T_3 \in \Delta \setminus \Delta'$ such that both the common edge e_1 of T_1, T_2 and the common edge e_2 of T_2, T_3 are degenerate at v (see Fig. 2.3, b)). We say that a vertex v is *semisingular* with respect to Δ' if it is semisingular of either type.

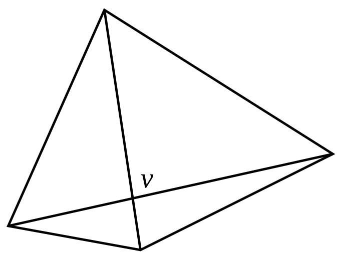


Fig. 2.1. A singular vertex.

a) b)

Fig. 2.2. Edge e is a) degenerate or b) nondegenerate at v.

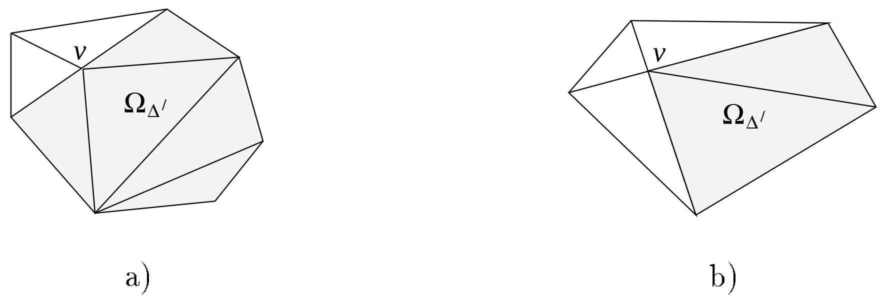


Fig. 2.3. Vertex v is semisingular w.r.t. Δ'.

Our analysis will involve second partial derivatives of $s_{|T}$, where $s \in S_q^1(\Delta)$ and $T \in \Delta$, at the vertices of the triangle T. While s is generally *not* twice differentiable at a vertex v, the following weaker condition always holds. Let $T', T'' \in \Delta$ have a common edge e attached to a vertex v and let r be the unit vector in the direction of e away from v. If $\hat{r}$ is another unit vector in the plane, which may, in particular, coincide with r, then

$$\frac{\partial^2(s_{|T'})}{\partial r \partial \hat{r}}(v) = \frac{\partial^2(s_{|T''})}{\partial r \partial \hat{r}}(v)\,. \tag{2.5}$$

Equation (2.5) immediately follows from the fact that

$$\frac{\partial(s_{|T'})}{\partial \hat{r}}(z) = \frac{\partial(s_{|T''})}{\partial \hat{r}}(z)\,, \quad \text{for any} \quad z \in e\,.$$

Following [15] we define “edge derivatives” of a function.

Definition 2.4 Suppose that $f \in C^1(\Omega)$ and $f_{|T} \in C^2(T)$, for any $T \in \Delta$. Let v be a vertex in Δ, let e_1, e_2 be two consecutive edges attached to v, and let T be the triangle with vertex v and edges e_1, e_2. By the *first*, respectively, *second* e_i*-derivative* of f at v we mean

$$\frac{\partial f}{\partial e_i}(v) := \frac{\partial f}{\partial r_i}(v) \quad \text{and} \quad \frac{\partial^2 f}{\partial e_i^2}(v) := \frac{\partial^2(f_{|T})}{\partial r_i^2}(v)\,, \qquad i = 1, 2\,, \tag{2.6}$$

where r_i is the unit vector in the e_i direction away from v. Furthermore, by the *cross* (e_1, e_2)*-derivative* of f at v we mean

$$\frac{\partial^2 f}{\partial e_1 \partial e_2}(v) := \frac{\partial^2(f_{|T})}{\partial r_1 \partial r_2}(v)\,. \tag{2.7}$$

In our construction of interpolation sets, some special subtriangulations play an essential role.

Definition 2.5 We say that $\Delta' \subset \Delta$ is a *tame subtriangulation* if the following conditions (T_1)–(T_3) hold:

(T$_1$) $\Omega_{\Delta'} := \bigcup_{T\in\Delta'} T$ is simply connected.

(T$_2$) For any two triangles $T', T'' \in \Delta'$, there exists a sequence $\{T_1, \ldots, T_\mu\} \subset \Delta'$ such that T_i and T_{i+1} have a common edge, $i = 1, \ldots, \mu - 1$, where $T_1 = T'$, $T_\mu = T''$.

(T$_3$) If two vertices $v_1, v_2 \in \Omega_{\Delta'}$ are connected by an edge e of the triangulation Δ, then $e \subset \Omega_{\Delta'}$.

It is easy to see that condition (T$_2$) in the above definition may be substituted by

(T$'_2$) $\Omega_{\Delta'}$ has a Jordan boundary.

Therefore, Δ is a tame subtriangulation of itself.

We note that a set $M \subset \mathbb{R}^2$ is simply connected if and only if it does not have holes, where by *hole* of M we mean any bounded connected component of $\mathbb{R}^2 \setminus M$. If M is connected but has holes, then obviously every such hole is simply connected.

Lemma 2.6 *Let Δ' be a tame subtriangulation of Δ, with $\Delta' \neq \Delta$. Then there exists a vertex $v \in \Omega \setminus \Omega_{\Delta'}$ such that*

$$\Omega_{\Delta'} \cap ST(v) \quad \text{includes an edge } e \text{ of the triangulation, and,} \tag{2.8}$$

$$\Omega_{\Delta'} \cup ST(v) \quad \text{is simply connected.} \tag{2.9}$$

Proof. Since Δ is a regular triangulation and Δ' is tame, it is easy to see that there exists at least one vertex $v_1 \in \Omega \setminus \Omega_{\Delta'}$ such that $\Omega_{\Delta'} \cap ST(v_1)$ includes an edge of the triangulation. If v_1 also satisfies (2.9), we set $v = v_1$. Otherwise, $\Omega_{\Delta'} \cup ST(v_1)$ has a hole H_1 with $\overline{H_1} = T_1 \cup \cdots \cup T_\mu$, where $T_j \in \Delta \setminus \Delta'$, $j = 1, \ldots, \mu$. Then $\Omega_{\Delta'} \cap \overline{H_1}$ evidently includes an edge e. Consider the triangle $T_{j_1} \subset \overline{H_1}$ attached to this edge. Since Δ' is tame, it follows from (T$_1$) and (T$_3$) that the third vertex of T_{j_1} cannot be in $\Omega_{\Delta'}$. (Indeed, if all three vertices of T_{j_1} were in $\Omega_{\Delta'}$, then by (T$_3$) also all three edges of T_{j_1} would be subsets of $\Omega_{\Delta'}$. Since $\Omega_{\Delta'}$ is simply connected by (T$_1$), we would have $T_{j_1} \subset \Omega_{\Delta'}$, a contradiction.) We denote by v_2 the vertex of T_{j_1} that does not lie in $\Omega_{\Delta'}$. If v_2 satisfies (2.9), we set $v = v_2$. Otherwise, $\Omega_{\Delta'} \cup ST(v_2)$ has a hole $H_2 \subset H_1$ (see Fig. 2.4). Since $T_{j_1} \subset \overline{H_1} \setminus H_2$, we have $H_2 \neq H_1$. Then we can find a triangle $T_{j_2} \subset \overline{H_2}$ attached to a common edge of $\Omega_{\Delta'}$ and $\overline{H_2}$, and denote by v_3 the vertex of of T_{j_2} that does not lie in $\Omega_{\Delta'}$. If v_3 satisfies (2.9), we set $v = v_3$. Otherwise, we proceed with this method and construct a sequence of vertices $v_1, v_2, \ldots \in \Omega \setminus \Omega_{\Delta'}$ satisfying (2.8) such that $\Omega_{\Delta'} \cup ST(v_j)$ has a hole H_j, $j = 1, 2, \ldots$, with

$$H_{j+1} \subset H_j, \quad H_{j+1} \neq H_j, \quad v_{j+1} \in \overline{H_j} \setminus \overline{H_{j+1}}, \qquad j = 1, 2, \ldots.$$

Since there is only a finite number of vertices in H_1, this process terminates after finitely many steps, and we finally obtain a vertex $v_k \in H_{k-1}$, $k \geq 1$, such that $\Omega_{\Delta'} \cup ST(v_k)$ has no hole, which means that $v = v_k$ satisfies (2.9). ■

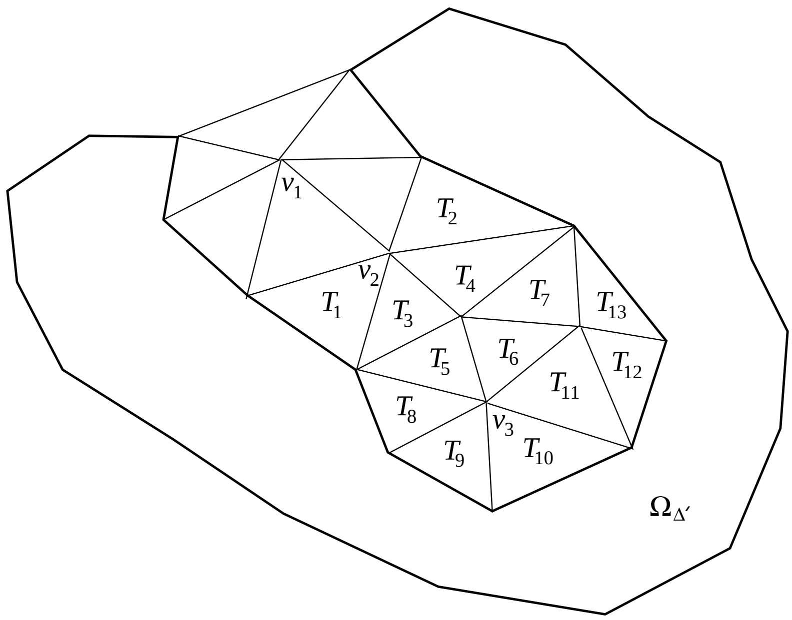


Fig. 2.4. Filling a hole: $v = v_3$, $\overline{H_1} = T_1 \cup \cdots T_{13}$, $\overline{H_2} = T_5 \cup \cdots T_{13}$.

3 Interpolation by C^1 Quartic Splines

We now construct a chain of subsets Ω_i of Ω such that

$$\emptyset = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m = \Omega\,,$$

and correspond to each Ω_i a set of points $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$, $i = 1, \ldots, m$.

For $i = 1$, we take $\Delta_1 = \{T_1\}$, where T_1 is an arbitrarily chosen "starting" triangle in Δ with vertices $v_1^{(1)}$, $v_1^{(2)}$ and $v_1^{(3)}$, and set

$$\Omega_1 := \Omega_{\Delta_1} = T_1\,.$$

We choose $\mathcal{L}_1^{(4)}$ to be an arbitrary set of 15 points lying on T_1 and admissible for Lagrange interpolation from Π_4. For example, we can choose three vertices $v_1^{(1)}, v_1^{(2)}, v_1^{(3)}$ of T_1, any three distinct points $w_{1,j}^{(1)}, w_{1,j}^{(2)}, w_{1,j}^{(3)}$ in the interior of the edge $e_{1,j}$, for each $j = 1, 2, 3$, where $e_{1,j}$ denotes the edge of T_1 which is opposite to the vertex $v_1^{(j)}$, and any three noncollinear points $z_1^{(1)}, z_1^{(2)}, z_1^{(3)}$ in the interior of the triangle T_1.

Proceeding by induction, we take $i \geq 2$ and suppose that Δ_{i-1} has already been defined and is a tame subtriangulation of Δ, with $\Omega_{i-1} := \Omega_{\Delta_{i-1}}$ being a proper subset of Ω. In order to construct Δ_i, we apply Lemma 2.6 to the subtriangulation Δ_{i-1} and choose a vertex $v_i \in \Omega \setminus \Omega_{i-1}$ such that $\Omega_{i-1} \cap ST(v_i)$ includes at least one edge and $\Omega_{i-1} \cup ST(v_i)$ is simply connected. It follows that there exists a sequence of (different) vertices $v_{i,0}, v_{i,1}, \ldots, v_{i,\mu_i}$, where $\mu_i \geq 1$, such that $v_{i,j-1}$ and $v_{i,j}$ are connected by an edge $e_{i,j}$ of the triangulation, $j = 1, \ldots, \mu_i$, and $\Omega_{i-1} \cap ST(v_i) = e_{i,1} \cup \cdots \cup e_{i,\mu_i}$. Denoting by $T_{i,j}$ the triangle with vertices v_i, $v_{i,j-1}$ and $v_{i,j}$, $j = 1, \ldots, \mu_i$, we set (see Fig. 3.1)

$$\Delta_i := \Delta_{i-1} \cup \{T_{i,1}, \ldots, T_{i,\mu_i}\} \quad \text{and} \quad \Omega_i := \Omega_{\Delta_i}\,.$$

It is easy to check that Δ_i satisfies (T$_1$)–(T$_3$), hence it is tame. In order to define $\mathcal{L}_i^{(4)}$, we need some further notation. Denote by $\hat{e}_{i,j}$ the edge attached to v_i and $v_{i,j}$, $j = 0, \ldots, \mu_i$, $i = 2, \ldots, m$ (see Fig. 3.2). For each $i \in \{2, \ldots, m\}$ we define $J_i \subset \{0, \ldots, \mu_i\}$ as follows:

$$(3.1) \qquad j \in J_i \quad \text{if and only if} \quad \begin{cases} \hat{e}_{i,j} \text{ is nondegenerate at } v_{i,j}, \text{ and} \\ \text{if } j \in \{0, \mu_i\}, \text{ then, in addition,} \\ v_{i,j} \text{ is semisingular w.r.t. } \Delta_i. \end{cases}$$

Moreover, for every $i \in \{2, \ldots, m\}$ we set

$$(3.2) \qquad \theta_i := \begin{cases} 1, & \text{if } v_i \text{ is semisingular w.r.t. } \Delta_i, \text{ but nonsingular,} \\ 0, & \text{otherwise.} \end{cases}$$

(Thus, $\theta_i = 1$ if and only if v_i is semisingular w.r.t. Δ_i and either $\hat{e}_{i,0}$ or $\hat{e}_{i,\mu_i}$ is nondegenerate at v_i.)

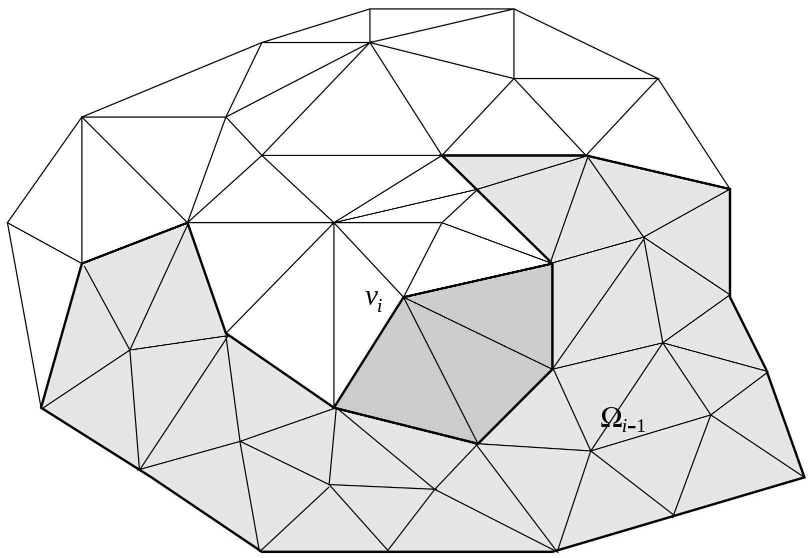


Fig. 3.1. Constructing Δ_i.

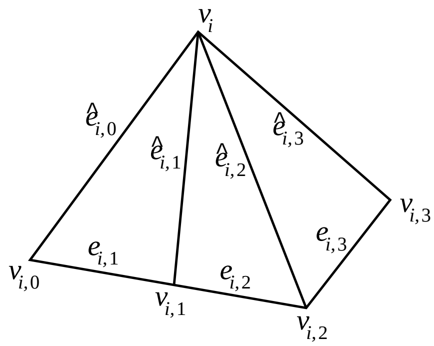


Fig. 3.2. $\Omega_i \setminus \Omega_{i-1}$ $(\mu_i = 3)$.

We distinguish three cases.

Case 1: $\theta_i = 0$.
Then $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of

(3.3) the vertex v_i,

(3.4) any point $w_{i,j}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus J_i$,

(3.5) two points w_i', w_i'', different from (3.4) and lying in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\}$, and

(3.6) any point z_i in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\}$.

Case 2: $\theta_i = 1$ and there exists $j^* \in \{0, \mu_i\} \setminus J_i$, such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i.
Then $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of (3.3),

(3.7) any point $w_{i,j}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus (J_i \cup \{j^*\})$,

(3.8) two points w_i', w_i'', different from (3.7) and lying in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\} \setminus \{j^*\}$, and

(3.9) any point z_i in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\} \setminus \{j^*, j^* + 1\}$.

(It is easy to see that such j', j'' and j''' exist.)

Case 3: $\theta_i = 1$ and $\hat{e}_{i,j}$ is degenerate at v_i for every $j \in \{0, \mu_i\} \setminus J_i$.

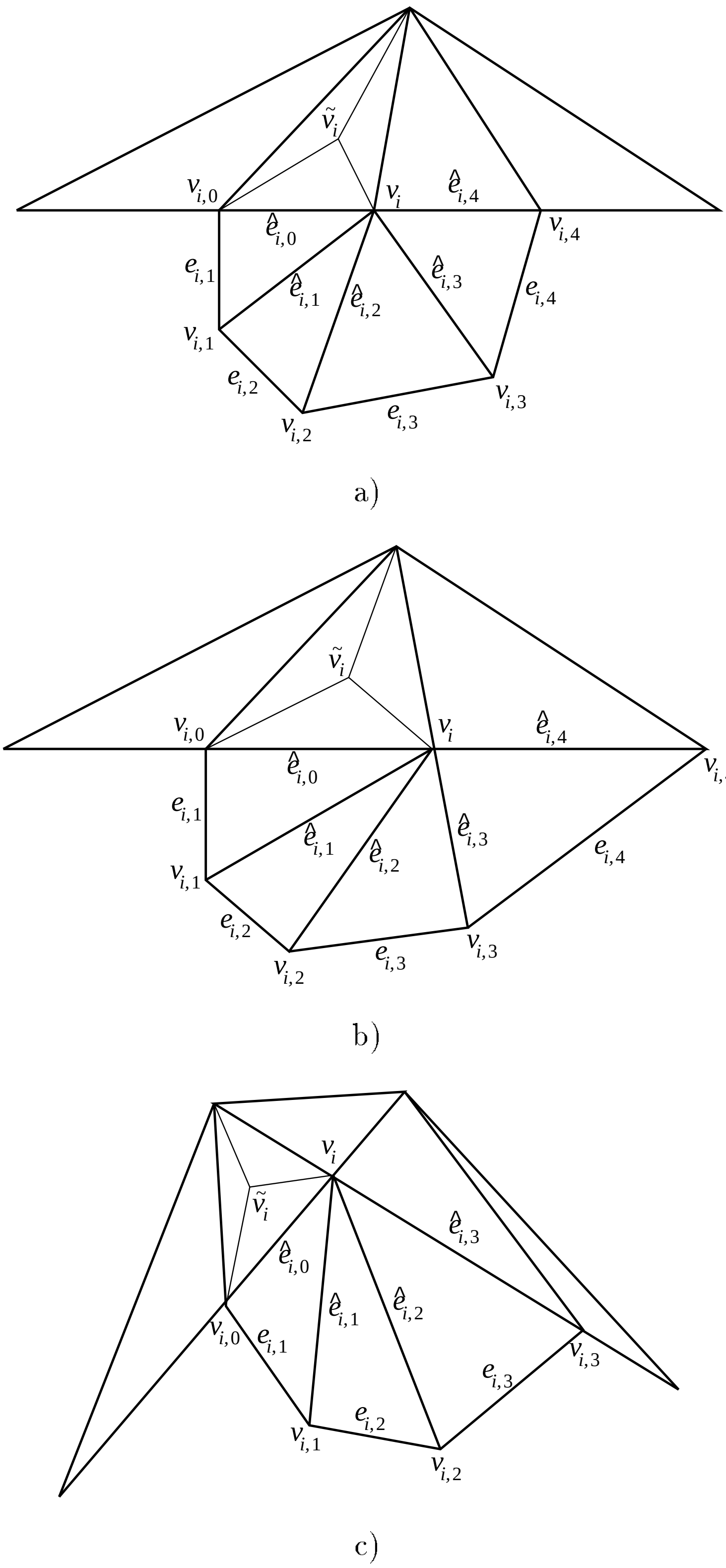


Fig. 3.3. Clough-Tocher split of a triangle in Case 3.

In this case we slightly modify the triangulation Δ locally as follows. The first possibility is to perform a Clough-Tocher split of the triangle $\tilde{T}_i$ that lies outside Ω_i and shares the edge $\hat{e}_{i,0}$ with $T_{i,1}$. Therefore, we add a new vertex $\tilde{v}_i$ in the interior of $\tilde{T}_i$ and connect $\tilde{v}_i$ with three edges to each of the vertices of $\tilde{T}_i$ (see Fig. 3.3). After this modification vertex v_i is no longer semisingular w.r.t. Δ_i, hence $\theta_i = 0$, and we choose $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$ according to the rule described in Case 1. Furthermore, we choose $v_{i+1} := \tilde{v}_i$. It is easy to see that $\Omega_i \cap ST(\tilde{v}_i)$ includes an edge of the triangulation, namely $\hat{e}_{i,0}$, and $\Omega_i \cup ST(\tilde{v}_i)$ is simply connected since $\Omega_{i-1} \cup ST(v_i)$ is simply connected. Therefore, Δ_{i+1} defined by adding to Δ_i the triangle with vertices v_i, $v_{i,0}$ and $\tilde{v}_i$, is a tame subtriangulation of Δ. Moreover, we have $\theta_{i+1} = 0$. Thus, we choose $\mathcal{L}_{i+1}^{(4)} \subset \Omega_{i+1} \setminus \Omega_i$ according to Case 1. The second possibility is to shift the vertex v_i so that θ_i becomes 0. Then we choose $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$ as in Case 1.

In both cases we denote the resulting modified triangulation by Δ^*.

Theorem 3.1 *The set of points $\mathcal{L}^{(4)} := \bigcup_{i=1}^{m} \mathcal{L}_i^{(4)}$ described above is a Lagrange interpolation set for $S_4^1(\Delta^*)$. In particular, $\Delta^* = \Delta$ if Case 3 does not occur.*

The proof of Theorem 3.1 will be given in Section 5.

Remark 3.2 (i) For $i = 2, \ldots, m$ the choice of the interpolation points in $\mathcal{L}_i^{(4)} \subset \Omega_i \setminus \Omega_{i-1}$ is based on the following consideration. (For details see the proof in Section 5.) Let $s \in S_4^1(\Delta)$ vanish on Ω_{i-1}. Since s is a C^1-spline, certain derivatives of s along the edges in $\Omega_i \setminus \Omega_{i-1}$ are implied to be zero, and certain sets of interpolation points can be chosen on $\Omega_i \setminus \Omega_{i-1}$. Now, what conditions are implied depends on nondegenerate edges and semisingular vertices in $\Omega_i \setminus \Omega_{i-1}$. This leads to the definition of the sets J_i in (3.1) and the indicator values θ_i in (3.2). Roughly speaking, nondegenerate edges and semisingular vertices in $\Omega_i \setminus \Omega_{i-1}$ reduce the number of interpolation points in $\mathcal{L}_i^{(4)}$.
(ii) We had to split the case $\theta_i = 1$ into two subcases (Case 2 and Case 3) depending on whether or not there exists $j^* \in \{0, \mu_i\} \setminus J_i$, such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i. If Case 3 is given, then, as said above, the triangulation Δ has to be slightly modified. This is necessary since in this case a spline in $S_4^1(\Delta_{i-1})$ that interpolates at the points of the set $\bigcup_{j=1}^{i-1} \mathcal{L}_j^{(4)}$ cannot in general be extended to a spline in $S_4^1(\Delta_i)$.

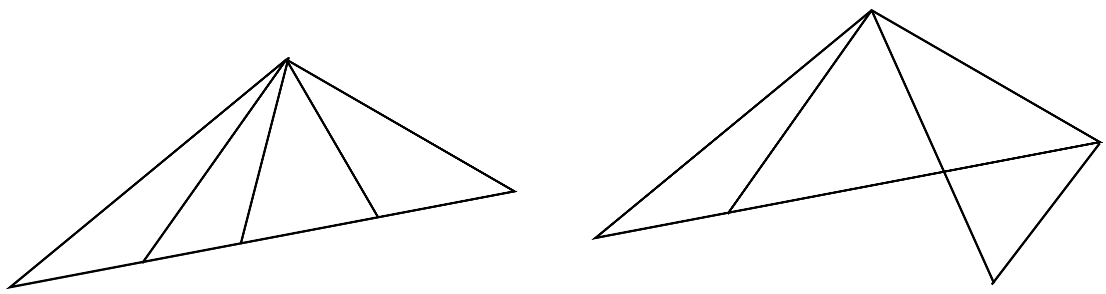
Remark 3.3 (i) We note that Case 3 is an exceptional case. It is easy to see that it may occur only in the following three specific situations:

- v_i is semisingular of type I w.r.t. Δ_i, and both $v_{i,0}$, v_{i,μ_i} are semisingular w.r.t. Δ_i, see Fig. 3.3, a),

- v_i is semisingular of type I w.r.t. Δ_i, one of two vertices $v_{i,j}$, $j = 0, \mu_i$, is semisingular w.r.t. Δ_i, and the other one is such that $\hat{e}_{i,j}$ is degenerate at v_i, see Fig. 3.3, b), or

- v_i is semisingular of type II w.r.t. Δ_i, and both $v_{i,0}$, v_{i,μ_i} are semisingular w.r.t. Δ_i, see Fig. 3.3, c).

(ii) In particular, the occurrence of Case 3 requires that Δ should include subtriangulations shown in Fig. 3.4, where a vertex is connected with five or four collinear vertices. Therefore, no modification of Δ is needed if each vertex is connected with at most three collinear vertices. In particular, this last condition is satisfied for any triangulation obtained from an arbitrary convex quadrangulation by inserting one or two diagonals of each quadrilateral.
(iii) We also note that our method works without modifying Δ if the total number of edges attached to v_i is odd. Then $\mathcal{L}_i^{(4)}$ consists of (3.3), (3.4) and (3.5) in Case 3.



a) b)

Fig. 3.4. Subtriangulations in Case 3.

Remark 3.4 (Hermite interpolation.) It is easy to see from the proof of Theorem 3.1, that Lagrange interpolation of f at some points of the above scheme can be replaced by interpolation of appropriate first or second partial derivatives of f provided that such derivatives exist. Namely, instead of interpolating function values at the points $w_{1,j}^{(1)}, w_{1,j}^{(3)}$, $j = 1, 2, 3$, one can require that

$$\frac{\partial s}{\partial x}(v_1^{(j)}) = \frac{\partial f}{\partial x}(v_1^{(j)}), \quad \frac{\partial s}{\partial y}(v_1^{(j)}) = \frac{\partial f}{\partial y}(v_1^{(j)}), \quad j = 1, 2, 3 .$$

For each $j = 1, 2, 3$, interpolation of f at $w_{1,j}^{(2)}$ can be replaced by the interpolation of second $e_{1,j}$-derivative of f at any of two vertices of $e_{1,j}$, and interpolation of f at $z_1^{(j)}$ can be replaced by the interpolation of cross derivative of f at $v_1^{(j)}$. For each $i = 2, \ldots, m$, interpolation of f at w_i', w_i'' can be replaced by the conditions

$$\frac{\partial s}{\partial x}(v_i) = \frac{\partial f}{\partial x}(v_i), \quad \frac{\partial s}{\partial y}(v_i) = \frac{\partial f}{\partial y}(v_i),$$

interpolation of f at $w_{i,j}$ can be replaced by

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_i) = \frac{\partial^2 f}{\partial \hat{e}_{i,j}^2}(v_i),$$

and interpolation of f at z_i can be replaced by

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j'''-1}\partial \hat{e}_{i,j'''}}(v_i) = \frac{\partial^2 f}{\partial \hat{e}_{i,j'''-1}\partial \hat{e}_{i,j'''}}(v_i)\,.$$

Particularly, our Hermite interpolation scheme includes the function and gradient values at all vertices of the triangulation.

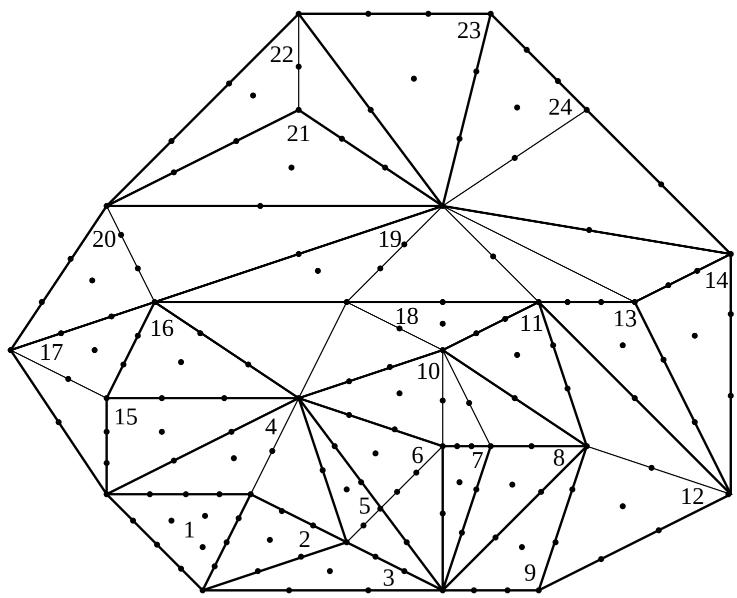


Fig. 3.5. Location of Lagrange interpolation points for $S_4^1(\Delta)$.
(Numbers and bold faced lines show the construction of $\{\Delta_i\}_{i=1}^m$.)

Remark 3.5 The computation of interpolating spline $s \in S_4^1(\Delta)$ according to our scheme is easy to perform step by step, by constructing $s_{|\Omega_i \setminus \Omega_{i-1}}$ after $s_{|\Omega_{i-1} \setminus \Omega_{i-2}}$. This can always be done by solving small systems of linear equations. In general, there exist much freedom in choosing the starting triangle T_1 as well as the vertices v_i, $i = 2, ..., m$, when constructing the chain of subtriangulations Δ_i with required properties. Therefore, in practice it is advantageous to take into account the special structure of the triangulation if it is known. For example, for three- and four-direction meshes our method leads to the interpolation schemes developed by Nürnberger [18] and Nürnberger & Walz [22], respectively. These schemes possess (nearly) optimal approximation order. (See also the numerical examples in [18] and [22].) In addition, our numerical experiments with Franke's test function on general classes of triangulations by using up to 40 000 interpolation conditions show that for $S_q^1(\Delta)$, $q \geq 4$, our methods also yield (nearly) optimal approximation order (see, *e.g.*, [10]). So far our numerical tests show that it is suitable to choose some of the Lagrange interpolation points near the vertices of Δ. If data are only given at the vertices, we approximatively compute the interpolation conditions needed for our algorithm by using local methods.

4 Interpolation by C^1 Splines of Degree $q \geq 5$

We construct a chain of subsets Ω_i of Ω such that

$$\emptyset = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m = \Omega\,,$$

as in Section 3 and assign to each Ω_i a set of points $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$, $i = 1, \ldots, m$, as follows. (In the case $q \geq 5$ no modification of the given triangulation Δ is necessary.)

We choose $\mathcal{L}_1^{(q)}$ to be an arbitrary set of d_q points lying on T_1 and admissible for Lagrange interpolation from Π_q. In order to define $\mathcal{L}_i^{(q)}$, $i \geq 2$, we distinguish two cases.

Case 1: $\theta_i = 0$.
Then $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of

(4.1) the vertex v_i,

(4.2) any $q-3$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-3)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus J_i$,

(4.3) any $q-4$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-4)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in J_i$,

(4.4) two points w_i', w_i'', different from (4.2) and (4.3) and lying in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\}$,

(4.5) any d_{q-4} distinct points $z_{i,j'''}^{(1)}, \ldots, z_{i,j'''}^{(d_{q-4})}$ lying in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\}$, and admissible for Lagrange interpolation from Π_{q-4}, and

(4.6) any d_{q-5} distinct points $z_{i,j}^{(1)}, \ldots, z_{i,j}^{(d_{q-5})}$ lying in the interior of $T_{i,j}$ and admissible for Lagrange interpolation from Π_{q-5}, for each $j \in \{1, \ldots, \mu_i\} \setminus \{j'''\}$.

Case 2: $\theta_i = 1$. (Hence, there exists $j^* \in \{0, \mu_i\}$, such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i.)
Then $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of (4.1),

(4.7) any $q-3$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-3)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus (J_i \cup \{j^*\})$, (4.8) any $q-4$ distinct points $w_{i,j}^{(1)},\ldots,w_{i,j}^{(q-4)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in J_i \setminus \{j^*\}$,

(4.9) any $q-\kappa$ distinct points $w_{i,j^*}^{(1)},\ldots,w_{i,j^*}^{(q-\kappa)}$ in the interior of the edge $\hat{e}_{i,j^*}$, where $\kappa = 5$ if $j^* \in J_i$, and $\kappa = 4$ if $j^* \notin J_i$,

(4.10) two points w_i', w_i'', different from (4.7), (4.8) and (4.9) and lying in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0,\ldots,\mu_i\} \setminus \{j^*\}$,

(4.11) any d_{q-4} distinct points $z_{i,j'''}^{(1)},\ldots,z_{i,j'''}^{(d_{q-4})}$ lying in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1,\ldots,\mu_i\}\setminus\{j^*, j^*+1\}$, and admissible for Lagrange interpolation from Π_{q-4}, and

(4.12) any d_{q-5} distinct points $z_{i,j}^{(1)},\ldots,z_{i,j}^{(d_{q-5})}$ lying in the interior of $T_{i,j}$ and admissible for Lagrange interpolation from Π_{q-5}, for each $j \in \{1,\ldots,\mu_i\} \setminus \{j'''\}$.

Theorem 4.1 *The set of points* $\mathcal{L}^{(q)} := \bigcup_{i=1}^m \mathcal{L}_i^{(q)}$ *described above is a Lagrange interpolation set for* $S_q^1(\Delta)$, $q \geq 5$.

Remark 4.2 In contrast to the algorithm of Section 3, we do not need to split the case $\theta_i = 1$ into subcases. The reason is that even if $\hat{e}_{i,j}$ is degenerate at v_i for every $j \in \{0,\mu_i\} \setminus J_i$, we can choose $j^* \in \{0,\mu_i\} \cap J_i$ such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i, which, by (4.9), simply reduces to $q-5$ the number of interpolation points to be taken in the interior of the edge $\hat{e}_{i,j^*}$. Therefore, $q-5$ must be nonnegative, which shows that this scheme is not applicable to $S_4^1(\Delta)$.

Remark 4.3 As in the case $q = 4$, the above Lagrange interpolation scheme can be transformed into an appropriate Hermite interpolation scheme for $S_q^1(\Delta)$, $q \geq 5$ (cp. Remark 3.4). Moreover, Remark 3.5 about computation and approximation order of our interpolation method remains true in the case $q \geq 5$.

The following result on the dimension of $S_q^1(\Delta)$, $q \geq 5$, restricted to a tame subtriangulation is a consequence of Theorem 4.1.

Corollary 4.4 *Let* Δ' *be a tame subtriangulation of* Δ. *Then*

$$\dim S_q^1(\Delta)_{|\Omega_{\Delta'}} = \dim S_q^1(\Delta') - \tilde{\sigma}(\Delta'), \quad q \geq 5, \tag{4.13}$$

where $\tilde{\sigma}(\Delta')$ *denotes the number of semisingular vertices w.r.t.* Δ' *which are nonsingular.*

The proofs of Theorem 4.1 and Corollary 4.4 will be given in Section 5.

5 Proofs

We prove Theorem 3.1 and Theorem 4.1 simultaneously and divide the proof into a series of lemmas.

First, we describe some relations between second derivatives of a C^1-spline on adjacent triangles. Note that (5.1) was given in [15].

Lemma 5.1 *Let $s \in S_q^1(\Delta)$, $q \geq 2$. Suppose that e', e, e'' are three consecutive edges attached to a vertex v. Denote by θ' and θ'' the angles between e and e' and between e and e'' respectively. Then*

$$\sin(\theta' + \theta'')\frac{\partial^2 s}{\partial e^2}(v) = \sin\theta''\frac{\partial^2 s}{\partial e \partial e'}(v) + \sin\theta'\frac{\partial^2 s}{\partial e \partial e''}(v)\,. \tag{5.1}$$

If, in addition, e' is degenerate at v and e^* denotes the edge attached to v which is opposite to e (see Fig. 5.1), then

$$\sin(\theta' + \theta'')\frac{\partial^2 s}{\partial e^2}(v) = -\sin\theta''\frac{\partial^2 s}{\partial e^* \partial e'}(v) + \sin\theta'\frac{\partial^2 s}{\partial e \partial e''}(v)\,. \tag{5.2}$$

Finally, if e^* is also degenerate at v, then

$$\sin(\theta' + \theta'')\frac{\partial^2 s}{\partial e^2}(v) = \sin\theta''\frac{\partial^2 s}{\partial e^* \partial e^{**}}(v) + \sin\theta'\frac{\partial^2 s}{\partial e \partial e''}(v)\,, \tag{5.3}$$

where e^{**} is the edge attached to v and opposite to e'.

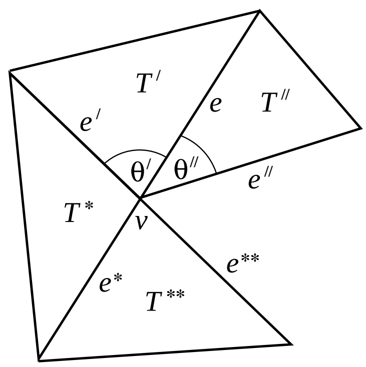


Fig. 5.1.

Proof. Denote by r, r', r'', r^* and r^{**} the unit vectors emerging from v in directions of e, e', e'', e^* and e^{**} respectively, and by T', T'', T^* and T^{**} the triangles formed by vertex v and pairs of edges e, e'; e, e''; e^*, e' and e^*, e^{**} respectively. Since

$$r\sin(\theta' + \theta'') = r'\sin\theta'' + r''\sin\theta'\,,$$

we have

$$\sin(\theta' + \theta'')\frac{\partial f}{\partial r}(v) = \sin\theta''\frac{\partial f}{\partial r'}(v) + \sin\theta'\frac{\partial f}{\partial r''}(v)\,,$$

for any differentiable at v function f. Therefore, by (2.5)–(2.7),

$$\begin{aligned}\sin(\theta' + \theta'')\frac{\partial^2 s}{\partial e^2}(v) &= \sin(\theta' + \theta'')\frac{\partial^2 (s_{|T'})}{\partial r^2}(v) \\ &= \sin\theta''\frac{\partial^2 (s_{|T'})}{\partial r \partial r'}(v) + \sin\theta'\frac{\partial^2 (s_{|T''})}{\partial r \partial r''}(v) \\ &= \sin\theta''\frac{\partial^2 s}{\partial e \partial e'}(v) + \sin\theta'\frac{\partial^2 (s)}{\partial e \partial e''}(v)\,,\end{aligned}$$

so that (5.1) holds. Furthermore, since $r^* = -r$, we have

$$\frac{\partial^2 s}{\partial e \partial e'}(v) = \frac{\partial^2 (s_{|T'})}{\partial r' \partial r}(v) = \frac{\partial^2 (s_{|T^*})}{\partial r' \partial r}(v) = -\frac{\partial^2 (s_{|T^*})}{\partial r' \partial r^*}(v) = -\frac{\partial^2 s}{\partial e^* \partial e'}(v)\,,$$

so that (5.2) follows from (5.1). Similarly, since $r^{**} = -r'$,

$$\frac{\partial^2 s}{\partial e^* \partial e'}(v) = \frac{\partial^2 (s_{|T^*})}{\partial r^* \partial r'}(v) = \frac{\partial^2 (s_{|T^{**}})}{\partial r^* \partial r'}(v) = -\frac{\partial^2 (s_{|T^{**}})}{\partial r^* \partial r^{**}}(v) = -\frac{\partial^2 s}{\partial e^* \partial e^{**}}(v)\,,$$

and (5.3) follows from (5.2). ∎

For any finite-dimensional linear space U of real-valued functions defined on a set Ω, let $u_{|\Omega'}$ denote the restriction of $u \in U$ to a subset Ω' of Ω. We set

$$U_{|\Omega'} := \{u_{|\Omega'} \;:\; u \in U\}\,,$$

$$U/\Omega' := \{u \in U \;:\; u_{|\Omega'} \equiv 0\}\,.$$

Suppose that a set of points $\mathcal{L}^{(q)} \subset \Omega$, $q \geq 4$, has been constructed according to the algorithm of Section 3 or 4. For $q = 4$ we may assume that Case 3 of the algorithm never occurs. Indeed, let Δ^* be the triangulation obtained from Δ by splitting some triangles $\tilde{T}_i$, respectively by shifting some vertices v_i, as described in Section 3. Then it is easy to see that $\mathcal{L}_i$ are chosen in the same way as if the algorithm were applied to Δ^* instead of Δ. (In fact, a difference could appear only if we split a triangle $\tilde{T}_i$ attached to the edge $\hat{e}_{i,0}$ such that $v_{i,0}$ is semisingular with respect to Δ_{i-1}. But then $\hat{e}_{i,0}$ is degenerate at $v_{i,0}$, hence $0 \notin J_i$ and $\hat{e}_{i,0}$ is nondegenerate at v_i, which never happens in Case 3. Therefore, we never split such a triangle.) Moreover, for Δ^* further modification is not needed since Case 3 does not occur. Thus, in the following we assume that Case 3 never occurs for Δ.

Consider the spaces

$$S_i^{(q)} := (S_q^1(\Delta)/\Omega_{i-1})_{|\Omega_i} \subset S_q^1(\Delta_i)\,, \qquad i = 1, \ldots, m\,. \tag{5.4}$$

Since $\Delta_0 = \emptyset$ and Δ_1 consists of a single triangle T_1, it is obvious that

$$S_1^{(q)} = \Pi_{q|_{T_1}}, \qquad \dim S_1^{(q)} = d_q\,. \tag{5.5}$$

For $i \geq 2$,

$$S_i^{(q)}{}_{|\Omega_{i-1}} \equiv 0\,, \quad S_i^{(q)}{}_{|\overline{\Omega_i \setminus \Omega_{i-1}}} \subset S_q^1(\Delta_i \setminus \Delta_{i-1}) = S_q^1(\{T_{i,1}, \ldots, T_{i,\mu_i}\})\,. \tag{5.6}$$

Lemma 5.2 *Every spline $s \in S_i^{(q)}$, $i \geq 2$, satisfies the following two conditions:*

$$s(z) = \frac{\partial s}{\partial x}(z) = \frac{\partial s}{\partial y}(z) = 0\,, \quad z \in e_{i,1} \cup \cdots \cup e_{i,\mu_i}\,, \tag{5.7}$$

and

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_{i,j}) = 0\,, \quad j \in J_i\,. \tag{5.8}$$

Proof. Given any spline $s \in S_i^{(q)}$, it follows from (5.6) that s satisfies (5.7). Making use of Lemma 5.1, we now prove that (5.8) also holds. First consider the case $j \in J_i \cap \{1, \ldots, \mu_i - 1\}$. By (5.7), we have

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial e_{i,j}}(v_{i,j}) = \frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial e_{i,j+1}}(v_{i,j}) = 0\,.$$

Then, by (5.1), with $e = \hat{e}_{i,j}$, $e' = e_{i,j}$ and $e'' = e_{i,j+1}$,

$$\sin(\theta' + \theta'')\frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_{i,j}) = \sin\theta'' \frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial e_{i,j}}(v_{i,j}) + \sin\theta' \frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial e_{i,j+1}}(v_{i,j}) = 0\,.$$

Since $\hat{e}_{i,j}$ is nondegenerate at $v_{i,j}$, we have $\sin(\theta' + \theta'') \neq 0$, which ensures (5.8). Let now $j \in J_i \cap \{0, \mu_i\}$, say $j = 0$. By the definition of J_i, the vertex $v_{i,0}$ is semisingular w.r.t. Δ_i. If it is semisingular of type I, then there exist two edges e', e^* attached to $v_{i,0}$ such that e' is degenerate and lies outside Ω_i, and e^* is opposite to $\hat{e}_{i,0}$ and lies on the boundary of Ω_{i-1}. Since $s_{|\Omega_{i-1}} \equiv 0$, we have

$$\frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial e_{i,1}}(v_{i,0}) = \frac{\partial^2 s}{\partial e^* \partial e'}(v_{i,0}) = 0\,.$$

Therefore, by (5.2),

$$\frac{\partial^2 s}{\partial \hat{e}_{i,0}^2}(v_{i,0}) = -\frac{\sin\theta''}{\sin(\theta' + \theta'')} \frac{\partial^2 s}{\partial e^* \partial e'}(v_{i,0}) + \frac{\sin\theta'}{\sin(\theta' + \theta'')} \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial e_{i,1}}(v_{i,0}) = 0\,,$$

and (5.8) holds. If now $v_{i,0}$ is semisingular of type II, then there exist three edges e', e^*, e^{**} attached to $v_{i,0}$ such that e' and e^* are degenerate and lie outside Ω_i, e^*

is opposite to $\hat{e}_{i,0}$ and e^{**} is opposite to e' and lies on the boundary of Ω_{i-1}. Since $s|_{\Omega_{i-1}} \equiv 0$, we have

$$\frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial e_{i,1}}(v_{i,0}) = \frac{\partial^2 s}{\partial e^* \partial e^{**}}(v_{i,0}) = 0\,.$$

Therefore, by (5.3),

$$\frac{\partial^2 s}{\partial \hat{e}_{i,0}^2}(v_{i,0}) = \frac{\sin\theta''}{\sin(\theta'+\theta'')}\frac{\partial^2 s}{\partial e^* \partial e^{**}}(v_{i,0}) + \frac{\sin\theta'}{\sin(\theta'+\theta'')}\frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial e_{i,1}}(v_{i,0}) = 0\,,$$

which completes the proof of the lemma. ■

Lemma 5.3 *Let T be a triangle with vertices $v^{(1)}$, $v^{(2)}$ and $v^{(3)}$. Denote by e_j the edge of T opposite to $v^{(j)}$, $j = 1, 2, 3$. Suppose that $p \in \Pi_4$ satisfies the following conditions*

$$p(z) \equiv 0\,, \quad z \in e_1 \cup e_2 \cup e_3\,,$$
$$\frac{\partial^2 p}{\partial e_3 \partial e_1}(v^{(2)}) = \frac{\partial^2 p}{\partial e_2 \partial e_1}(v^{(3)}) = 0\,,$$

and either

$$\frac{\partial^2 p}{\partial e_3 \partial e_2}(v^{(1)}) = 0\,,$$

or there exists a point z' in the interior of T, such that

$$p(z') = 0\,.$$

Then p is the zero function.

Proof. First we show that

$$\frac{\partial p}{\partial e_2}(z) \equiv 0\,, \quad z \in e_1\,. \tag{5.9}$$

Indeed, $\tilde{p} := \left(\frac{\partial p}{\partial e_2}\right)|_{e_1}$ is a univariate polynomial of degree 3, such that

$$\tilde{p}(v^{(3)}) = \tilde{p}'(v^{(3)}) = \tilde{p}(v^{(2)}) = 0\,.$$

Moreover, $\tilde{p}'(v^{(2)}) = \frac{\partial^2 p}{\partial e_1 \partial e_2}(v^{(2)})$ is a linear combination of $\frac{\partial^2 p}{\partial e_1 \partial e_3}(v^{(2)})$ and $\frac{\partial^2 p}{\partial e_1^2}(v^{(2)})$, both being zero. Hence $\tilde{p}'(v^{(2)}) = 0$, which proves (5.9).

For each $j = 1, 2, 3$, let

$$l_j(z) = 0$$

be the equation of the straight line containing the edge e_j. Since $p(z) \equiv 0$ for all $z \in e_1 \cup e_2 \cup e_3$, and (5.9) holds, it follows that

$$p(z) \equiv \lambda(l_1(z))^2 l_2(z) l_3(z)\,,$$

where λ is a real constant. (See, for example, [3, p. 42].)

It is easy to see that either of the two conditions

$$\frac{\partial^2 p}{\partial e_3 \partial e_2}(v^{(1)}) = 0 \quad \text{or} \quad p(z') = 0$$

implies

$$\lambda = 0\,,$$

which completes the proof of the lemma. ■

Lemma 5.4 *For each $i = 1, \ldots, m$, $\mathcal{L}_i^{(q)}$ is a total set with respect to $S_i^{(q)}$.*

Proof. We first consider the case $q = 4$ in details. For $i = 1$ the assertion follows from the fact that $\mathcal{L}_1^{(4)}$ is admissible for Lagrange interpolation from Π_4. Let $i \geq 2$. Since $\theta_i \in \{0, 1\}$, we have two cases.

Case 1: $\theta_i = 0$.

Let $s \in S_i^{(4)}$, and

$$\begin{aligned} s(v_i) = s(w_i') = s(w_i'') = s(z_i) = 0\,, \\ s(w_{i,j}) = 0 \quad \text{for all } j \in \{0, \ldots, \mu_i\} \setminus J_i\,, \end{aligned} \tag{5.10}$$

where the points v_i, w_i', w_i'', z_i and $w_{i,j}$ are chosen according to (3.3)–(3.6). We have to show that

$$p_j := s_{|T_{i,j}} \equiv 0\,, \quad j = 1, \ldots, \mu_i\,.$$

To this end we will check that the assumptions of Lemma 5.3 are satisfied for each p_j on the corresponding triangle $T_{i,j}$.

By Lemma 5.2, the spline s satisfies (5.7) and (5.8). It follows from (5.7) that

$$p_j(z) \equiv 0\,, \quad z \in e_{i,j}\,, \quad j = 1, \ldots, \mu_i\,,$$

$$\frac{\partial^2 p_j}{\partial e_{i,j} \partial \hat{e}_{i,j-1}}(v_{i,j-1}) = \frac{\partial^2 p_j}{\partial e_{i,j} \partial \hat{e}_{i,j}}(v_{i,j}) = 0\,, \quad j = 1, \ldots, \mu_i\,.$$

Moreover, $\tilde{p}_j := s_{|\hat{e}_{i,j}}$ is a univariate polynomial of degree 4 such that, by (5.7) and (5.8),

$$\begin{aligned} \tilde{p}_j(v_{i,j}) = \tilde{p}_j'(v_{i,j}) = 0\,, \quad j = 0, \ldots, \mu_i\,, \\ \tilde{p}_j''(v_{i,j}) = 0\,, \quad j \in J_i\,, \end{aligned}$$

and, by (5.10),

$$\begin{aligned} \tilde{p}_j(w_{i,j}) = 0\,, \quad j \in \{0, \ldots, \mu_i\} \setminus J_i\,, \\ \tilde{p}_j(v_i) = 0\,, \quad j = 0, \ldots, \mu_i\,, \\ \tilde{p}_{j'}(w_i') = \tilde{p}_{j''}(w_i'') = 0\,, \end{aligned}$$

from which it immediately follows that $\tilde{p}_{j'} \equiv 0$ and $\tilde{p}_{j''} \equiv 0$. Then, since s is differentiable at v_i, we have

$$\frac{\partial s}{\partial x}(v_i) = \frac{\partial s}{\partial y}(v_i) = 0\,,$$

and, hence,

$$\tilde{p}_j'(v_i) = 0\,, \quad j = 0, \ldots, \mu_i\,.$$

Therefore, it follows that $\tilde{p}_j \equiv 0$ for all $j = 0, \ldots, \mu_i$. Going back to the bivariate polynomials p_j, we have

$$p_j(z) \equiv 0\,, \quad z \in \hat{e}_{i,j-1} \cup \hat{e}_{i,j}\,, \quad j = 1, \ldots, \mu_i\,.$$

Since $p_{j'''}(z_i) = 0$, it follows from Lemma 5.3 that

$$p_{j'''} \equiv 0\,.$$

Lemma 5.3 can also be applied to the other polynomials p_j if we show that

$$\frac{\partial^2 p_j}{\partial \hat{e}_{i,j-1} \partial \hat{e}_{i,j}}(v_i) = 0\,, \quad j = 1, \ldots, \mu_i\,, \quad j \neq j'''\,. \tag{5.11}$$

It follows from (5.1) that for $j = 1, \ldots, \mu_i - 1$,

$$\sin(\theta_j + \theta_{j+1}) \frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_i) = \sin\theta_{j+1} \frac{\partial^2 p_j}{\partial \hat{e}_{i,j} \partial \hat{e}_{i,j-1}}(v_i) + \sin\theta_j \frac{\partial^2 p_{j+1}}{\partial \hat{e}_{i,j} \partial \hat{e}_{i,j+1}}(v_i)\,,$$

where θ_j is the angle between $\hat{e}_{i,j}$ and $\hat{e}_{i,j-1}$. Since $s(z) = 0$ for all $z \in \hat{e}_{i,j}$, we have

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_i) = 0\,, \quad j = 0, \ldots, \mu_i\,.$$

Then it follows from the above equations that

$$\begin{aligned} \frac{1}{\sin\theta_1} \frac{\partial^2 p_1}{\partial \hat{e}_{i,0} \partial \hat{e}_{i,1}}(v_i) &= -\frac{1}{\sin\theta_2} \frac{\partial^2 p_2}{\partial \hat{e}_{i,1} \partial \hat{e}_{i,2}}(v_i) \\ &= \cdots = (-1)^{\mu_i - 1} \frac{1}{\sin\theta_{\mu_i}} \frac{\partial^2 p_{\mu_i}}{\partial \hat{e}_{i,\mu_i-1} \partial \hat{e}_{i,\mu_i}}(v_i)\,. \end{aligned}$$

However, we have already proved that $p_{j'''} \equiv 0$, hence

$$\frac{\partial^2 p_{j'''}}{\partial \hat{e}_{i,j'''-1} \partial \hat{e}_{i,j'''}}(v_i) = 0\,,$$

and (5.11) follows.

Case 2: $\theta_i = 1$.

Then there exists $j^* \in \{0, \mu_i\}$, for which $\hat{e}_{i,j^*}$ is nondegenerate at v_i. Moreover, by construction, we may assume that $j^* \notin J_i$. Therefore, $\mathcal{L}_i^{(4)}$ is defined by (3.3), (3.7), (3.8) and (3.9). Suppose $s \in S_i^{(4)}$ satisfies

$$\begin{aligned} s(v_i) = s(w_i') = s(w_i'') = s(z_i) = 0\,, \\ s(w_{i,j}) = 0\,, \quad \text{for all } j \in \{0, \ldots, \mu_i\} \setminus J_i\ ,\ \ j \neq j^*\,. \end{aligned} \tag{5.12}$$

If we show that

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j^*}^2}(v_i) = 0\,, \tag{5.13}$$

then the proof will proceed exactly as in Case 1, except that (5.13) will be used instead of $s(w_{i,j^*}) = 0$. In order to prove (5.13), we first deduce from (5.1) that

$$\begin{aligned} \sin(\theta_j + \theta_{j+1}) \frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_i) = \\ \sin\theta_{j+1} \frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial \hat{e}_{i,j-1}}(v_i) + \sin\theta_j \frac{\partial^2 s}{\partial \hat{e}_{i,j} \partial \hat{e}_{i,j+1}}(v_i)\,, \quad j = 1, \ldots, \mu_i - 1\,. \end{aligned} \tag{5.14}$$

Moreover, since $\theta_i = 1$, v_i is semisingular with respect to Δ_i. If it is semisingular of type I, then, by (5.1), we get

$$\sin(\theta_1 + \theta') \frac{\partial^2 s}{\partial \hat{e}_{i,0}^2}(v_i) = \sin\theta' \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial \hat{e}_{i,1}}(v_i) + \sin\theta_1 \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial \hat{e}'}(v_i)\,,$$

and, by (5.2),

$$\sin(\theta' - \theta_{\mu_i}) \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i}^2}(v_i) = \sin\theta' \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i} \partial \hat{e}_{i,\mu_i - 1}}(v_i) - \sin\theta_{\mu_i} \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial \hat{e}'}(v_i)\,,$$

where e' is the edge attached to v_i such that e' does not lie in Ω_i, and θ' is the angle between $\hat{e}_{i,0}$ and e'. Therefore,

$$\begin{aligned} \frac{\sin(\theta_1 + \theta')}{\sin\theta' \sin\theta_1} \frac{\partial^2 s}{\partial \hat{e}_{i,0}^2}(v_i) + \frac{\sin(\theta' - \theta_{\mu_i})}{\sin\theta' \sin\theta_{\mu_i}} \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i}^2}(v_i) = \\ \frac{1}{\sin\theta_1} \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial \hat{e}_{i,1}}(v_i) + \frac{1}{\sin\theta_{\mu_i}} \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i} \partial \hat{e}_{i,\mu_i - 1}}(v_i)\,. \end{aligned} \tag{5.15}$$

Similarly, if v_i is semisingular of type II w.r.t. Ω_i, then, by (5.1) and (5.3), we have

$$\begin{aligned} \frac{\sin(\theta_1 + \theta')}{\sin\theta' \sin\theta_1} \frac{\partial^2 s}{\partial \hat{e}_{i,0}^2}(v_i) - \frac{\sin(\theta' + \theta_{\mu_i})}{\sin\theta' \sin\theta_{\mu_i}} \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i}^2}(v_i) = \\ \frac{1}{\sin\theta_1} \frac{\partial^2 s}{\partial \hat{e}_{i,0} \partial \hat{e}_{i,1}}(v_i) - \frac{1}{\sin\theta_{\mu_i}} \frac{\partial^2 s}{\partial \hat{e}_{i,\mu_i} \partial \hat{e}_{i,\mu_i - 1}}(v_i)\,, \end{aligned} \tag{5.16}$$

where θ' is the angle between $\hat{e}_{i,0}$ and the edge e' attached to v_i such that e' does not lie in Ω_i and is not opposite to $\hat{e}_{i,0}$. On the other hand, it follows from (5.7), (5.8) and (5.12) that

$$s(z) \equiv 0\,, \quad z \in \hat{e}_{i,j}\,, \quad \text{for each} \quad j \in \{0,\ldots,\mu_i\} \setminus \{j^*\}\,,$$
$$s(z) \equiv 0\,, \quad z \in T_{i,j'''}\,.$$

Therefore, we have

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_i) = 0\,, \quad \text{for all} \quad j \in \{0,\ldots,\mu_i\} \setminus \{j^*\}\,,$$
$$\frac{\partial^2 s}{\partial \hat{e}_{i,j'''-1}\partial \hat{e}_{i,j'''}}(v_i) = 0\,,$$

which, together with (5.14)–(5.16), imply (5.13).

For $q \geq 5$ the proof is similar. For example, in Case 2 the assumption that $s \in S_i^{(q)}$ satisfies $s(z) = 0$ for all points z listed in (4.1), (4.7)–(4.12) implies, in view of Lemma 5.2, that

$$s(z) = \tfrac{\partial s}{\partial x}(z) = \tfrac{\partial s}{\partial y}(z) = 0\,, \quad z \in e_{i,1} \cup \cdots \cup e_{i,\mu_i}\,,$$
$$s(z) = 0\,, \quad z \in \hat{e}_{i,j}\,, \quad \text{for each} \quad j \in \{0,\ldots,\mu_i\} \setminus \{j^*\}\,,$$
$$s(v_i) = \tfrac{\partial s}{\partial \hat{e}_{i,j^*}}(v_i) = s(v_{i,j^*}) = \tfrac{\partial s}{\partial \hat{e}_{i,j^*}}(v_{i,j^*}) = 0\,,$$
$$s(w_{i,j^*}^{(1)}) = \cdots = s(w_{i,j^*}^{(q-\kappa)}) = 0\,,\ \kappa \in \{4,5\}\,, \quad \tfrac{\partial^2 s}{\partial \hat{e}_{i,j}^2}(v_{i,j^*}) = 0 \text{ if } \kappa = 5\,,$$
$$s(z_{i,j}^{(1)}) = \cdots = s(z_{i,j}^{(d_{q-5})}) = 0\,, \quad \text{for each} \quad j \in \{1,\ldots,\mu_i\} \setminus \{j'''\}\,,$$
$$s(z_{i,j'''}^{(1)}) = \cdots = s(z_{i,j'''}^{(d_{q-4})}) = 0\,.$$

Therefore,

$$s_{|T_{i,j'''}}(z) \equiv p(z)(l_1(z))^2 l_2(z) l_3(z)\,,$$

where $l_j(z) = 0$, $j = 1,2,3$ are the equations of the straight lines containing $e_{i,j'''}$, $\hat{e}_{i,j'''}$ and $\hat{e}_{i,j'''-1}$, respectively, and $p(z)$ is a polynomial in Π_{q-4}. Then we have

$$p(z_{i,j'''}^{(1)}) = \cdots = p(z_{i,j'''}^{(d_{q-4})}) = 0\,,$$

and, hence, $p(z) \equiv 0$. Thus, we get

$$s(z) \equiv 0\,, \quad z \in T_{i,j'''}\,,$$

which, in view of (5.14)–(5.16), implies

$$\frac{\partial^2 s}{\partial \hat{e}_{i,j^*}^2}(v_i) = 0\,.$$

Therefore,

$$s(z) = 0\,, \quad z \in \hat{e}_{i,j^*}\,,$$

and, by induction, starting from $T_{i,j'''}$, we show that

$$s_{|_{T_{i,j}}}(z) \equiv 0\,, \quad \text{for all } j = 0, \ldots, \mu_i\,. \quad \blacksquare$$

Lemma 5.5 *$\mathcal{L}^{(q)} = \cup_{i=1}^m \mathcal{L}_i^{(q)}$ is a total set with respect to $S_q^1(\Delta)$.*

Proof. Suppose that $s \in S_q^1(\Delta)$ vanishes at all points $z \in \mathcal{L}^{(q)}$. Then $s(z) = 0$ for all $z \in \mathcal{L}_1^{(q)}$, and by Lemma 5.4, we get $s_{|_{\Omega_1}} \equiv 0$. We proceed by induction on i. Assume that for some $i \in \{2, \ldots, m\}$,

$$s_{|_{\Omega_{i-1}}} \equiv 0\,.$$

Then, by the definition of $S_i^{(q)}$, we have $s_{|_{\Omega_i}} \in S_i^{(q)}$. Since $s(z) = 0$ for all $z \in \mathcal{L}_i^{(q)}$, it follows from Lemma 5.4 that

$$s_{|_{\Omega_i}} \equiv 0\,.$$

Since $\Omega_m = \Omega$, the assertion follows. $\blacksquare$

Since every total set of points whose cardinality coincides with the dimension of the space is a Lagrange interpolation set, both Theorems 3.1 and 4.1 will be established if we prove that

$$\text{card}\,\mathcal{L}^{(q)} = \dim S_q^1(\Delta)\,. \tag{5.17}$$

To this end we need the following lemma.

Lemma 5.6 *The following inequality holds*

$$\sum_{i=2}^m \text{card}\, J_i \geq V_I - \sigma - \sum_{i=2}^m \theta_i\,. \tag{5.18}$$

Proof. Since $\sum_{i=2}^m \theta_i$ is exactly the number of nonsingular interior vertices v_i, for which $\theta_i = 1$, it suffices to prove that all other nonsingular interior vertices belong to the set

$$\bigcup_{i=2}^m \mathcal{J}_i\,,$$

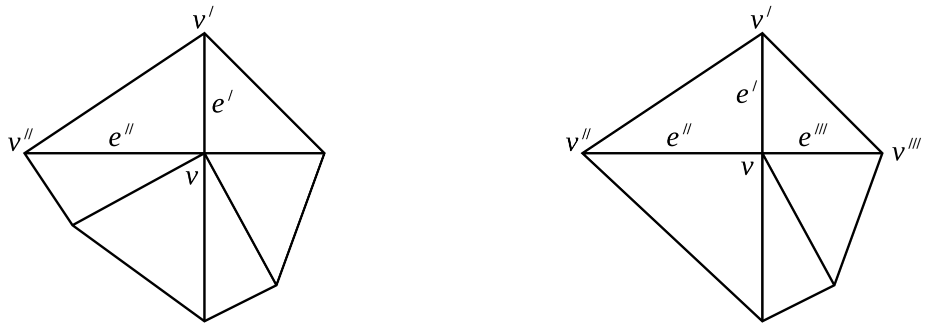
where

$$\mathcal{J}_i := \{v_{i,j} : j \in J_i\}\,, \quad i = 2, \ldots, m\,.$$

Indeed, then the total number of nonsingular interior vertices of Δ, *i.e.*, $V_I - \sigma$, does not exceed

$$\text{card}\left(\bigcup_{i=2}^m \mathcal{J}_i\right) + \sum_{i=2}^m \theta_i \leq \sum_{i=2}^m \text{card}\, J_i + \sum_{i=2}^m \theta_i\,,$$

and (5.18) follows. Let v be a nonsingular interior vertex of Δ such that either v belongs to the starting triangle T_1 or $v = v_i$ for some $i \geq 2$, with $\theta_i = 0$. Consider the set of all vertices lying on $ST(v)$ and assume that there are at least three of them that do not belong to T_1 (the cases when there are only one or two vertices in $ST(v) \setminus T_1$ can be treated in the same way). Let $v', v'', v''' \in ST(v)$ be such that $v' = v_{i'}$, $v'' = v_{i''}$ and $v''' = v_{i'''}$, with $i' > i'' > i'''$, and any other vertex $v^* \in ST(v)$ either belongs to T_1 or $v^* = v_{i^*}$ with $2 \leq i^* < i'''$. Thus, v''', v'' and v' were added the latest among all the vertices lying on $ST(v)$, when the construction of the chain of subtriangulations Δ_i was being performed. It is clear that $v \neq v'$ since otherwise $\Omega_{i'-1}$ would include all vertices of $ST(v)$ except v itself which is not possible for a tame subtriangulation. If now the edge e' between v and v' is nondegenerate at v, then $v \in \mathcal{J}_{i'}$. Suppose that e' is degenerate at v. Then we consider v''. If $v'' = v$, then $v = v_i$ is semisingular of type I w.r.t. Ω_i and $\theta_i = 1$ in contrast to our assumption. Therefore $v'' \neq v$. Since $\Omega_{i''-1}$ is tame, v, v' and v'' must share a triangle. (See Fig. 5.2, a).) If the edge e'' between v and v'' is nondegenerate at v, then $v \in \mathcal{J}_{i''}$ because v is semisingular of type I w.r.t. $\Omega_{i''}$. Suppose e'' is degenerate at v. Then we consider v'''. If $v''' = v$, then $v = v_i$ would be semisingular of type II w.r.t. Ω_i and $\theta_i = 1$. Therefore $v''' \neq v$. Since $\Omega_{i'''-1}$ is tame, v''' and v must share a triangle either with v' or with v''. (See Fig. 5.2, b).) In both cases the edge e''' between v and v''' is nondegenerate at v for otherwise v would be singular. Then v is semisingular of type II w.r.t. $\Omega_{i'''}$ and $v \in \mathcal{J}_{i'''}$. Thus, in either case $v \in \bigcup_{i=2}^{m} \mathcal{J}_i$, which completes the proof of the lemma. ■



a) b)

Fig. 5.2. Vertex v belongs to $\mathcal{J}_{i''}$ or $\mathcal{J}_{i'''}$.

It is easy to see that

$$\text{(5.19)} \qquad \operatorname{card} \mathcal{L}_i^{(q)} = \begin{cases} d_q\,, & \text{if } i = 1\,, \\ d_{q-4}\mu_i + 2q - 3 - \theta_i - \operatorname{card} J_i\,, & \text{if } i = 2, \dots, m\,. \end{cases}$$

By the construction of $\{\Delta_i\}_{i=1}^m$,

$$\begin{aligned} m &= V_I + V_B - 2\,, \\ \sum_{i=2}^m \mu_i &= N - 1\,. \end{aligned}$$

Therefore, by (5.19), (2.2), Lemma 5.6 and (2.3),

$$\begin{aligned} \operatorname{card}\mathcal{L}^{(q)} &= \sum_{i=1}^m \operatorname{card}\mathcal{L}_i^{(q)} = d_q + \sum_{i=2}^m (d_{q-4}\mu_i + 2q - 3 - \theta_i - \operatorname{card} J_i) \\ &= d_q + d_{q-4}(N-1) + (2q-3)(V_I + V_B - 3) - \sum_{i=2}^m (\theta_i + \operatorname{card} J_i) \\ &= d_q + d_{q-2}E_I - (d_q - d_1)V_I + V_I - \sum_{i=2}^m (\theta_i + \operatorname{card} J_i) \\ &\le d_q + d_{q-2}E_I - (d_q - d_1)V_I + \sigma = \dim S_q^1(\Delta)\,. \end{aligned}$$

This implies that $\operatorname{card}\mathcal{L}^{(q)} = \dim S_q^1(\Delta)$ since the cardinality of a total set can never be less than dimension, which shows (5.17) and completes the proof of Theorems 3.1 and 4.1.

Proof of Corollary 4.4. We first show that for all $q \ge 5$,

$$\dim S_i^{(q)} = \operatorname{card}\mathcal{L}_i^{(q)}\,, \quad i = 1, \ldots, m\,. \tag{5.20}$$

Indeed, by Lemma 5.4 $\mathcal{L}_i^{(q)}$ is a total set w.r.t. $S_i^{(q)}$, which implies

$$\dim S_i^{(q)} \le \operatorname{card}\mathcal{L}_i^{(q)}\,.$$

On the other hand, $S_i^{(q)} = (S_q^1(\Delta)/\Omega_{i-1})_{|\Omega_i}$ (see (5.4)). Then by Theorem 4.1,

$$\begin{aligned} \sum_{i=1}^m \dim S_i^{(q)} &= \sum_{i=1}^m \left(\dim S_q^1(\Delta)/\Omega_{i-1} - \dim S_q^1(\Delta)/\Omega_i\right) \\ &= \dim S_q^1(\Delta) = \sum_{i=1}^m \operatorname{card}\mathcal{L}_i^{(q)}\,, \end{aligned}$$

which can only be true if (5.20) holds.

Let now Δ' be an arbitrary tame subtriangulation of Δ. It is easy to see that the chain of subsets $\Omega_i \subset \Omega$, $i = 1, \ldots, m$, as in Section 3, can be chosen in such a way that

$$\emptyset = \Omega_0 \subset \Omega_1 \subset \cdots \subset \Omega_{m'} = \Omega_{\Delta'} \subset \Omega_{m'+1} \subset \cdots \subset \Omega_m = \Omega\,,$$

where $m'+2$ is the number of vertices in Δ'. Then

$$\begin{aligned}\sum_{i=1}^{m'} \dim S_i^{(q)} &= \sum_{i=1}^{m'} \left(\dim S_q^1(\Delta)/\Omega_{i-1} - \dim S_q^1(\Delta)/\Omega_i\right) \\ &= \dim S_q^1(\Delta) - \dim S_q^1(\Delta)/\Omega_{m'} = \dim S_q^1(\Delta)_{|\Omega_{\Delta'}}\,.\end{aligned}$$

Therefore, in view of (5.20), (5.19) and (2.2),

$$\begin{aligned}\dim S_q^1(\Delta)_{|\Omega_{\Delta'}} &= \sum_{i=1}^{m'} \operatorname{card} \mathcal{L}_i^{(q)} = d_q + \sum_{i=2}^{m'} (d_{q-4}\mu_i + 2q - 3 - \theta_i - \operatorname{card} J_i) \\ &= d_q + d_{q-2}E_I' - (d_q - d_1)V_I' + V_I' - \sum_{i=2}^{m'} (\theta_i + \operatorname{card} J_i)\,,\end{aligned}$$

where E_I' and V_I' denote the number of interior edges and vertices of Δ', respectively.

On the other hand, as in the proof of Theorem 4.1, we have

$$\dim S_q^1(\Delta') = d_q + d_{q-2}E_I' - (d_q - d_1)V_I' + V_I' - \sum_{i=2}^{m'} (\theta_i' + \operatorname{card} J_i')\,,$$

where J_i' and θ_i' are defined by (3.1) and (3.2), respectively, in regard to Δ' instead of Δ. Thus, it remains to show that

$$\sum_{i=2}^{m'} (\theta_i + \operatorname{card} J_i) = \sum_{i=2}^{m'} (\theta_i' + \operatorname{card} J_i') + \tilde{\sigma}(\Delta')\,. \tag{5.21}$$

To this end we consider the following sets of vertices of Δ':

$$\begin{aligned}\mathcal{V} &:= \{v_i \,:\, 2 \le i \le m' \text{ and } \theta_i = 1\}\,, \\ \mathcal{V}' &:= \{v_i \,:\, 2 \le i \le m' \text{ and } \theta_i' = 1\}\,, \\ \mathcal{J}_i &:= \{v_{i,j} : j \in J_i\}\,, \quad i = 2, \ldots, m'\,, \\ \mathcal{J}_i' &:= \{v_{i,j} : j \in J_i'\}\,, \quad i = 2, \ldots, m'\,.\end{aligned}$$

It is easy to check that $\mathcal{V}' \subset \mathcal{V}$ and $\mathcal{J}_i' \subset \mathcal{J}_i$, $i = 2, \ldots, m'$. Moreover, the sets

$$\mathcal{V} \setminus \mathcal{V}'\,, \quad \mathcal{J}_i \setminus \mathcal{J}_i'\,,\ i = 2, \ldots, m'\,,$$

are pairwise disjoint, and $(\mathcal{V} \setminus \mathcal{V}') \cup \bigcup_{i=2}^{m'} (\mathcal{J}_i \setminus \mathcal{J}_i')$ coincides with the set of all semisingular vertices w.r.t. Δ' which are nonsingular. Therefore,

$$\begin{aligned}\sum_{i=2}^{m'} (\theta_i + \operatorname{card} J_i) - \sum_{i=2}^{m'} (\theta_i' + \operatorname{card} J_i') &= \operatorname{card} \mathcal{V} \setminus \mathcal{V}' + \sum_{i=2}^{m'} \operatorname{card} \mathcal{J}_i \setminus \mathcal{J}_i' \\ &= \operatorname{card} \left((\mathcal{V} \setminus \mathcal{V}') \cup \bigcup_{i=2}^{m'} (\mathcal{J}_i \setminus \mathcal{J}_i') \right) = \tilde{\sigma}(\Delta')\,,\end{aligned}$$

and (5.21) is proved. ■

Acknowledgment. The authors would like to thank the referees for their valuable suggestions.

References

[1] M. H. Adam, *Bivariate Spline-Interpolation auf Crosscut-Partitionen*, Dissertation, Mannheim, 1995.

[2] P. Alfeld, B. Piper and L. L. Schumaker, An explicit basis for C^1 quartic bivariate splines, SIAM J. Numer. Anal. **24** (1987), 891–911.

[3] C. K. Chui, "Multivariate splines," CBMS-SIAM, Philadelphia, 1988.

[4] C. K. Chui and T. X. He, On location of sample points in C^1 quadratic bivariate spline interpolation, in L. Collatz, G. Meinardus, G. Nürnberger (eds.), *Numerical Methods of Approximation Theory, ISNM 81*, Birkhäuser, Basel, 1987, 30–43.

[5] C. K. Chui and D. Hong, Construction of local C^1 quartic spline elements for optimal-order approximation, Math. Comp. **65** (1996), 85–98.

[6] C. K. Chui and D. Hong, Swapping edges of arbitrary triangulations to achieve the optimal order of approximation, SIAM J. Numer. Anal. **34** (1997), 1472–1482.

[7] O. Davydov, On almost interpolation, J. Approx. Theory **91** (1997), 398–418.

[8] O. Davydov, Locally linearly independent basis for C^1 bivariate splines of degree $q \geq 5$, in M. Daehlen, T. Lyche, L. L. Schumaker (eds.), *Mathematical Methods for Curves and Surfaces II*, Vanderbilt University Press, 1998, 71–78.

[9] O. Davydov, G. Nürnberger and F. Zeilfelder, Approximation order of bivariate spline interpolation for arbitrary smoothness, J. Comput. Appl. Math. **90** (1998), 117–134.

[10] O. Davydov, G. Nürnberger and F. Zeilfelder, Interpolation by splines on triangulations, in *New Developments in Approximation Theory*, M. D. Buhmann, M. Felten, D. H. Mache, and M. W. Müller (eds.), ISNM, Birkhäuser, to appear.

[11] O. Davydov, M. Sommer and H. Strauss, On almost interpolation by multivariate splines, in *Multivariate Approximation and Splines*, G. Nürnberger, J. W. Schmidt, and G. Walz (eds.), ISNM, Birkhäuser, 1997, 45–58.

[12] O. Davydov, M. Sommer and H. Strauss, Locally linearly independent systems and almost interpolation, in *Multivariate Approximation and Splines*, G. Nürnberger, J. W. Schmidt, and G. Walz (eds.), ISNM, Birkhäuser, 1997, 59–72.

[13] O. Davydov, M. Sommer and H. Strauss, On almost interpolation and locally linearly independent bases, East J. Approx. **5** (1999), 67–88.

[14] F. Jeeawock-Zedek, Interpolation scheme by C^1 cubic splines on a non uniform type-2 triangulation of a rectangular domain, C.R. Acad. Sci. Ser. I Math., **314** (1992), 413–418.

[15] J. Morgan and R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$, Math. Comp. **29** (1975), 736–740.

[16] E. Nadler, Hermite interpolation by C^1 bivariate splines, *in* "Contributions to the computation of curves and surfaces," (W. Dahmen, M. Gasca and C. Micchelli, Eds.), Academia de Ciencias, Zaragoza, 1990, 55–66.

[17] G. Nürnberger, "Approximation by Spline Functions," SpringerVerlag, Berlin, Heidelberg, New York, 1989.

[18] G. Nürnberger, Approximation order of bivariate spline interpolation, *J. Approx. Theory* **78** (1996), 117–136.

[19] G. Nürnberger, O. Davydov, G. Walz and F. Zeilfelder, Interpolation by bivariate splines on crosscut partitions, *in* "Multivariate Approximation and Splines," (G. Nürnberger, J. W. Schmidt and G. Walz, Eds.), ISNM, Birkhäuser, 1997, 189–203.

[20] G. Nürnberger and Th. Riessinger, Lagrange and Hermite interpolation by bivariate splines, Numer. Func. Anal. Optim. **13** (1992), 75–96.

[21] G. Nürnberger and Th. Riessinger, Bivariate spline interpolation at grid points, Numer. Math. **71** (1995), 91–119.

[22] G. Nürnberger and G. Walz, Error analysis in interpolation by bivariate C^1-splines, IMA J. Numer. Anal. **18** (1998), to appear.

[23] M. Sommer and H. Strauss, Interpolation by uni- and multivariate generalized splines, J. Approx. Theory **83** (1995), 423–447.

[24] M. Sommer and H. Strauss, A condition of Schoenberg-Whitney type for multivariate spline interpolation, Advances in Comp. Math. **5** (1996), 381–397.

[25] Z. Sha, On interpolation by $S_3^1(\Delta_{m,n}^2)$, Approx. Theory Appl. **1** (1985), 1–18.

[26] Z. Sha, On interpolation by $S_2^1(\Delta_{m,n}^2)$, Approx. Theory Appl. **1** (1985), 71–82.

Oleg Davydov
Justus-Liebig-Universität
Mathematisches Institut
D-35392 Giessen, Germany
`oleg.davydov@math.uni-giessen.de`

Günther Nürnberger
Universität Mannheim
Fakultät für Mathematik und Informatik
68131 Mannheim, Germany
`nuern@euklid.math.uni-mannheim.de`