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SUMMARY

A New Version of Quantitative Resonance Principle and Applications

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1. Introduction

Let X be a Banach space with norm $\|\cdot\|_X$ and $\{p_n\}_{n=1}^{\infty}$ be a sequence of continuous sublinear functionals (seminorms) on X. According to the classical uniform boundedness principle in its "resonance" form, if there exists a sequence $\{f_n\}_{n=1}^{\infty} \subset X$ with $\|f_n\|_X \leqslant 1$ and $\limsup_{n\to\infty} p_n(f_n) = \infty$, then an individual element $f_* \in X$ satisfying $\limsup_{n\to\infty} p_n(f_*) = \infty$ also exists. It is evident that f_* can be chosen so that it satisfies $\|f_*\|_X \leqslant 1$.

We can say that the uniform boundedness principle partially answers the following general question. What conditions on $\mathfrak{M} \subset X$ and $\{f_n\} \subset \mathfrak{M}$ provide the existence of an individual element $f_* \in \mathfrak{M}$ such that $p_n(f_*)$ has the same asymptotic behavior as $p_n(f_n)$? Undoubtedly, "the same asymptotic behavior" can be taken variously. In the above-mentioned case of classical resonance principle it means that $p_n(f_*)$ is unbounded provided $p_n(f_n)$ is unbounded. But it is often required to construct an element $f_* \in \mathfrak{M}$ for which a more precise condition, for example

$$\limsup_{n\to\infty} p_n(f_*)/p_n(f_n) > 0 \ , \qquad \limsup_{n\to\infty} p_n(f_*)/p_n(f_n) \geqslant 1 \ ,$$

$$\liminf_{n\to\infty} p_n(f_*)/p_n(f_n) > 0 \ , \qquad \liminf_{n\to\infty} p_n(f_*)/p_n(f_n) \geqslant 1$$

or $\lim_{n\to\infty} p_n(f_*)/p_n(f_n) = 1$ holds.

If we choose a sequence $\{ f_n \}_{n=1}^{\infty} \subset \mathfrak{M}$ such that $p_n(f_n) = = p_n(\mathfrak{M}) := \sup \{ p_n(f) : f \in \mathfrak{M} \}$ or $p_n(f_n) \geqslant p_n(\mathfrak{M})(1-\varepsilon_n)$, $\varepsilon_n \to 0$, then our problem leads to the second question. What conditions on $\mathfrak{M}$ provide the existence of an individual element $f_* \in \mathfrak{M}$ such that $p_n(f_*)$ has the same asymptotic behavior as $p_n(\mathfrak{M})$?

Problems of this type arise, for example, in approximation theory, and many specific results have been obtained in the case that $\mathfrak{M}$ is a class of differentiable functions and p_n is the functional of the best approximation or the error functional of a linear method of approximation (see [19]).

A general approach to constructing elements $f_* \in \mathfrak{M}$ satisfying

$$\limsup_{n\to\infty} p_n(f_*)/p_n(f_n) > 0 \tag{1}$$

was developed by W. Dickmeis and R. J. Nessel (see [16]). They unified the gliding hump method arguments previously used by many authors for solving such problems in various specific cases. E. van Wickeren [23] presented an alternative approach to the subject based on considering $\mathfrak{M}$ as a Frechet space and applying Baire's category theorem. His method makes possible proving residuality of the sets of elements satisfying (1) and, in some instances, obtaining more precise results concerned with existence of elements $f_* \in \mathfrak{M}$ satisfying

$$\limsup_{n\to\infty} p_n(f_*)/p_n(f_n) \geqslant 1 \tag{2}$$

and residuality of the sets of such elements.

A new version of this "quantitative resonance principle" was worked out by the author in 1987 (see [4, 12]). At the time, however, we were not conversant with van Wickeren's result. So, our proof originally made use of the gliding hump method and, therefore, did not allow to deduce the residuality (this flaw was recently remedied in [14]). Nevertheless, our conditions on $\mathfrak{M}$ and $\{ f_n \} \subset \mathfrak{M}$ providing the existence of $f_* \in \mathfrak{M}$ satisfying (2) seem to be weaker and more easily verified. Because of this, our result found many applications, primarily, in approximation theory and made it possible to answer some questions posed by P. L. Butzer and W. Dickmeis, N. P. Korneichuk, K. I. Oskolkov.

In what follows we give the statement of our version of quantitative resonance principle and describe its applications and some related results.

2. Main Theorem and Its Consequences

A closed, absolutely convex set $\mathfrak{M}$ in a Banach space X can be defined by

$$\mathfrak{M} = \mathfrak{M}_H := \left\{ f \in X : \sup_{h \in H} h(f) \leqslant 1 \right\},$$

where H is a family of continuous seminorms. Denote

$$\mathfrak{M}_H^* := \left\{ f \in \mathfrak{M}_H : \lim_{\|h\| \to \infty} h(f) = 0 \right\}$$

($\|h\| = \sup \{ h(f) : f \in X , \|f\|_X \leqslant 1 \}$).

Theorem 2.1. [12] *Let $\{ p_n \}_{n=1}^{\infty}$ be a sequence of continuous seminorms on X, $\{ f_n \}_{n=1}^{\infty}$ be a sequence of elements $f_n \in \mathfrak{M}_H$ such that $p_n(f_n) \neq 0$, $n \in \mathbb{N}$. If*

$$f_n \in \mathfrak{M}_H^* , \quad n \in \mathbb{N} \tag{3}$$

and

$$\lim_{n \to \infty} \|f_n\|_X = 0 , \tag{4}$$

then there exists an element $f_* \in \mathfrak{M}_H$ satisfying (2).

The hypotheses (3) and (4) can be changed for some conditions on $\mathfrak{M}_H$, as the next two statements show (see [12]).

Corollary 2.2. *If* $\mathfrak{M}_H^*$ *is everywhere dense in* $\mathfrak{M}_H$ *(in the metric space* X*), then Theorem* 2.1 *holds true without supposition* (3).

Corollary 2.3. *Let* $\mathfrak{M}_H$ *be compact. Then under the hypotheses of Theorem* 2.1 *or Corollary* 2.2, *apart from* (4), *there exists an element* $f_* \in \mathfrak{M}_H$ *satisfying*

$$\limsup_{n\to\infty} p_n(f_*)/p_n(f_n) \geqslant 1/3. \tag{5}$$

Let us put

$$p_n(\mathfrak{M}) := \sup_{f\in\mathfrak{M}} p_n(f) \ .$$

The case that

$$\lim_{n\to\infty} \ p_n(f_n)/p_n(\mathfrak{M}) = 1$$

is of particular interest for the applications in approximation theory. Some general "resonance" results on asymptotic behavior of $p_n(\mathfrak{M})$ follow immediately from Corollary 2.3.

Corollary 2.4. [12] *Suppose* $\mathfrak{M}^*_H$ *is everywhere dense in* $\mathfrak{M}_H$ *and* $\mathfrak{M}_H$ *is compact. If* $\{ p_n \}_{n=1}^{\infty}$ *is a sequence of continuous seminorms on* X *with* $p_n(\mathfrak{M}_H) \neq 0$, $n \in \mathbb{N}$, *then there exists an element* $f_* \in \mathfrak{M}_H$ *such that*

$$\limsup_{n\to\infty} p_n(f_*)/p_n(\mathfrak{M}_H) \geqslant 1/3. \tag{6}$$

Corollary 2.5. [12] *Under the hypotheses of Corollary* 2.4, *the following properties of a sequence of seminorms* p_n *are equivalent:*

(a) $\lim_{n\to\infty} p_n(f) = 0$, $\forall f \in \mathfrak{M}_H$,

(b) $\lim_{n\to\infty} p_n(\mathfrak{M}_H) = 0$.

The following consequence of Theorem 2.1 gives an answer to the question in what cases $h(f)$ reaches its maximal possible value 1 for an element $f_* \in \mathfrak{M}_H$ and for whole sequence of seminorms $h \in H$.

Corollary 2.6. [15] *Let* H *be an unbounded family of continuous seminorms. Assume that the set*

$$X_H^* = \left\{ f \in X : \lim_{\|h\|\to\infty} h(f) = 0 \right\} = \bigcup_{\alpha>0} \alpha\, \mathfrak{M}_H^*$$

is everywhere dense in X. *Then there exists an element* $f_* \in \mathfrak{M}_H$ *such that*

$$\limsup_{\|h\|\to\infty} h(f_*) = 1 . \tag{7}$$

It follows from the main result in [14] that under the hypotheses of Theorem 2.1 the set of elements $f_* \in \mathfrak{M}_H$ satisfying (2) is residual in $\mathfrak{M}_H$. Because of this Corollaries 2.2 - 2.4, 2.6 as well as their consequences in what follows, are valid for residual sets of elements f_*.

It is also shown in [14] that no result on residuality is possible if we try to construct an element $f_* \in \mathfrak{M}$ satisfying

$$\liminf_{n\to\infty} p_n(f_*)/p_n(\mathfrak{M}) > 0 . \tag{8}$$

More precisely, the following result has been obtained.

Theorem 2.7. [14] *Let* $\{ p_n \}_{n=1}^{\infty}$ *be a sequence of continuous seminorms on a Banach space* X, $\mathfrak{M}$ *be a subset in* X. *If there exists a subset* $Q \subset \mathfrak{M}$ *being everywhere dense in* $\mathfrak{M}$ *such that*

$$\lim_{n\to\infty} p_n(f)/p_n(\mathfrak{M}) = 0 \quad , \quad \forall f \in Q , \tag{9}$$

then the set

$$\left\{ f \in \mathfrak{M} : \liminf_{n\to\infty} p_n(f_*)/p_n(\mathfrak{M}) > 0 \right\}$$

is of first Baire category.

3. Applications to Semigroup Operators and Peetre's K-functional

Assume that $T = \{ T(t) : t \geqslant 0 \}$ is a (C_0)-semigroup on a Banach space X. P.L. Butzer and W. Dickmeis [1] posed the problem of presenting a criterion which delivers the existence of non-optimal rates of convergence for semigroup operators, i.e. the existence of elements $f_\alpha \in X$ such that

$$\|T(t)f_\alpha - f_\alpha\|_X \begin{cases} = O(t^\alpha) \\ \neq o(t^\alpha) \end{cases} \qquad (t \to 0+) \tag{10}$$

for each $0 < \alpha < 1$. They showed that non-optimal rates of convergence exist if the infinitesimal generator of T possesses a sequence of eigenvalues λ_n with $\lim_{n\to\infty} |\lambda_n| = \infty$ and noted that it seemed plausible that (10) holds true for any semigroup T having unbounded generator.

In [15] we validated this conjecture by presenting the following more general and more precise result.

The modulus of continuity related to a (C_0)-semigroup T on X is defined by

$$\omega_T(f; t) = \sup_{0<h\leq t} \|T(h)f - f\|_X , \quad f \in X , \quad t > 0 .$$

Let

$$\mathfrak{M}_\omega(T) = \left\{ f \in X : \forall t \in [0, 1] \quad \omega_T(f; t) \leqslant \omega(t) \right\} ,$$

where $\omega(t)$ is a positive continuous function on $[0;1]$ with

$$\lim_{t\to 0+} \omega(t) = 0 . \tag{11}$$

As a consequence of Corollary 2.6 we have

Theorem 3.1. [15] *Suppose T is a (C_0)-semigroup with unbounded generator, $\omega(t)$ is a positive continuous function on $[0, 1]$ satisfying both (11) and*

$$\lim_{t\to 0+} \omega(t)/t = \infty . \tag{12}$$

Then there exists an element $f_* \in \mathfrak{M}_\omega(T)$ *such that*

$$\limsup_{t\to 0+} \omega_T(f_*;t)/\omega(t) = 1.$$

If we set $\omega(t) = t^\alpha$, $0 < \alpha < 1$, then obviously f_* satisfies (10).

Theorem 2.1 as well as Corollaries 2.2 - 2.5 can also be applied to classes $\mathfrak{M}_\omega(T)$. The following result was obtained in [15].

THEOREM 3.2. *Let* T *be a* (C_0)*-semigroup on* X*. Suppose that* $\omega(t)$ *is a positive non-decreasing continuous function on* $[0,1]$ *satisfying* (11), (12) *and*

$$\omega(t'')/t'' \leqslant K\,\omega(t')/t' \quad , \quad 0 < t' < t'' \leqslant 1 \tag{13}$$

with a constant $K > 0$. *Let* $\{p_n\}_{n=1}^{\infty}$ *be a sequence of continuous seminorms on* X.

(a) *If a sequence* $\{f_n\}_{n=1}^{\infty} \subset \mathfrak{M}_\omega(T)$ *satisfies* (4) *and* $p_n(f_n) \neq 0$, $n \in \mathbb{N}$, *then there exists an element* $f_* \in \mathfrak{M}_\omega(T)$ *such that* (2) *holds.*

(b) *If* $\mathfrak{M}_\omega(T)$ *is compact and* $p_n(\mathfrak{M}_\omega(T)) \neq 0$, $n \in \mathbb{N}$, *then there exists an element* $f_* \in \mathfrak{M}_\omega(T)$ *such that*

$$\limsup_{n\to\infty} p_n(f_*)/p_n(\mathfrak{M}_\omega(T)) \geqslant 1/3.$$

(c) *If* $\mathfrak{M}_\omega(T)$ *is compact, then* $\lim\limits_{n\to\infty} p_n(\mathfrak{M}_\omega(T)) = 0$ *if and only if* $\forall f \in \mathfrak{M}_\omega(T)$ $\lim\limits_{n\to\infty} p_n(f) = 0$.

Similar results hold true for the sets defined through the Peetre's K-functional.

Let U be a linear manifold in Banach space X with $|\cdot|_U$ is a given seminorm on U. The K-functional is defined by

$$K_t(f) = K_t(f;X;U) = \inf\left\{ \|f-g\|_X + t|g|_U : g \in U \right\}, \quad f \in X .$$

For a given positive continuous function $\omega(t)$ satisfying (11) we consider the set

$$X_\omega(U) = \left\{ f \in X : K_t(f) \leqslant \omega(t) \ , \quad t>0 \right\} .$$

Theorem 3.3. [12] *Suppose $\omega(t)$ is a positive non-decreasing continuous function on $[0,\infty)$ satisfying (11), (12) and (13). Let $\{ p_n \}_{n=1}^{\infty}$ be a sequence of continuous seminorms on X.*

(a) If a sequence $\{ f_n \}_{n=1}^{\infty} \subset X_\omega(U)$ satisfies (4) and $p_n(f_n) \neq 0$, $n \in \mathbb{N}$, then there exists an element $f_* \in X_\omega(U)$ such that (2) holds.

(b) If $X_\omega(U)$ is compact and $p_n(X_\omega(U)) \neq 0$, $n \in \mathbb{N}$, then there exists an element $f_* \in X_\omega(U)$ such that

$$\limsup_{n\to\infty} p_n(f_*)/p_n(X_\omega(U)) \geqslant 1/3.$$

(c) If $X_\omega(U)$ is compact, then $\lim\limits_{n\to\infty} p_n(X_\omega(U)) = 0$ if and only if $\forall f \in X_\omega(U)$ $\lim\limits_{n\to\infty} p_n(f) = 0$.

Theorem 3.4. [15] *Suppose U is everywhere dense in X, $\omega(t)$ is a positive continuous function on $[0,\infty)$ satisfying (11) and (12). If the condition*

$$\limsup_{t\to 0+} \|K_t\|/\omega(t) = \infty$$

holds, then there exists an element $f_* \in X_\omega(U)$ such that

$$\limsup_{t\to 0+} K_t(f_*)/\omega(t) = 1.$$

4. Sets $X(\varepsilon)$.

Let $\{ F_n \}_{n=1}^{\infty}$ be a sequence of subspaces in a Banach space X with $F_n \subset F_{n+1}$, $n \in \mathbb{N}$ and $\bigcup\limits_{n\in\mathbb{N}} F_n$ is everywhere dense in

X . Denote $E(f,F_n)_X = \inf \{ \|f-g\|_X : g \in F_n \}$, $E(\mathfrak{M},F_n)_X = \sup \{ E(f,F_n)_X : f \in \mathfrak{M} \}$, $\mathfrak{M} \subset X$. For every positive non-decreasing number sequence $\{ \varepsilon_n \}_{n=1}^{\infty}$ with $\lim_{n\to\infty} \varepsilon_n = 0$, we define

$$X(\varepsilon) = \left\{ f \in X : E(f,F_n)_X \leqslant \varepsilon_n \, , \; n \in \mathbb{N} \right\} .$$

The following result was obtained in [9] through the use of Theorem 2.1.

THEOREM 4.1. *Let* $\{ p_n \}_{n=1}^{\infty}$ *be a sequence of continuous seminorms on* X.

(a) *If a sequence* $\{ f_n \}_{n=1}^{\infty} \subset X(\varepsilon)$ *satisfies* (4) *and* $p_n(f_n) \neq 0$, $n \in \mathbb{N}$, *then there exists an element* $f_* \in X(\varepsilon)$ *such that* (2) *holds.*

(b) *If all* F_n , $n \in \mathbb{N}$, *are finite-dimensional spaces and* $p_n(X(\varepsilon)) \neq 0$, $n \in \mathbb{N}$, *then there exists an element* $f_* \in X(\varepsilon)$ *such that*

$$\limsup_{n\to\infty} p_n(f_*)/p_n(X(\varepsilon)) \geqslant 1/3. \tag{14}$$

(c) *If all* F_n , $n \in \mathbb{N}$, *are finite-dimensional spaces , then* $\lim_{n\to\infty} p_n(X(\varepsilon)) = 0$ *if and only if* $\forall f \in X(\varepsilon)$ $\lim_{n\to\infty} p_n(f) = 0$.

Now we describe a special case of sequence $\{ p_n \}_{n=1}^{\infty}$ for which it is possible to replace $1/3$ by 1 in the righthand side of (14).

Let $U : X \longrightarrow X$ be a bounded linear operator. Denote

$$e(\mathfrak{M},U)_X = \sup \left\{ \|f - U(f)\|_X : f \in \mathfrak{M} \right\} , \quad \mathfrak{M} \subset X .$$

The operator U is said to be exact in $\mathfrak{M}$ if $e(\mathfrak{M},U)_X = 0$.

THEOREM 4.2. [9] *Let* $\{ U_n : n \in \mathbb{N} \}$ *be a sequence of bounded linear operators mapping the Banach space* X *into itself with*

$e(X(\varepsilon), U_n)_X \neq 0$, $n \in \mathbb{N}$. If for each $m \in \mathbb{N}$ there exists a number $N = N(m)$ such that all U_n , $n \geq N$, are exact in F_m , then there exists an element $f_* \in X(\varepsilon)$ satisfying

$$\limsup_{n\to\infty} \|f_* - U_n(f_*)\|_X \,/\, e(X(\varepsilon), U_n)_X = 1 . \qquad (15)$$

In the specific case $X = C$, the Banach space of all 2π-periodic continuous functions with the uniform norm $\|f\|_C = \max\{ |f(t)| : t \in \mathbb{R} \}$, and $F_n = T_n$, the space of trigonometric polynomials of degree at most n , $n = 0, 1, \dots$, Theorem 4.2 can be applied to the investigation of the asymptotic behavior of the remainder of approximation of functions $f \in C(\varepsilon)$ by de la Vallée Poussin sums

$$\sigma_{np}(f) := \frac{1}{p+1} \sum_{\nu=n-p}^{n} s_\nu(f) , \quad 0 \leq p \leq n , \quad n = 0, 1, \dots$$

(where $s_\nu(f)$ is the ν-th partial sum of the Fourier series of f).

COROLLARY 4.3. [9] *Assume that $0 \leq p_n \leq n$, $n = 0, 1, \dots$ and $\lim_{n\to\infty} (n - p_n) = \infty$. Then there exists a function $f_* \in C(\varepsilon)$ such that*

$$\limsup_{n\to\infty} \|f_* - \sigma_{n,p_n}(f_*)\|_C \,/\, e(C(\varepsilon), \sigma_{n,p_n})_C = 1 .$$

Setting $p_n = 0$, $n = 0, 1, \dots$ we deduce a partial answer to a problem raised by K.I.Oskolkov [20] who has shown that

$$A_1 \sum_{\nu=0}^{n} \varepsilon_{n+\nu}/(\nu+1) \leq e(C(\varepsilon), s_n)_C \leq A_2 \sum_{\nu=0}^{n} \varepsilon_{n+\nu}/(\nu+1)$$

and noted that the question of the existence of an (independent of n) function $f \in C(\varepsilon)$ such that

$$\|f - s_n(f)\|_C \geq A_3 \sum_{\nu=0}^{n} \varepsilon_{n+\nu}/(\nu+1) , \quad n = 0, 1, \dots,$$

remained open.

In [10] we managed to construct such a function by applying a specific argument not using Theorem 2.1. Thus, the following statement holds.

THEOREM 4.4. [10] *There exists a function* $f_* \in C(\varepsilon)$ *such that*

$$\liminf_{n\to\infty} \|f_* - s_n(f_*)\|_C \,/\, e(C(\varepsilon),\, s_n)_C \;>\; 0\,.$$

A multidimensional extension of this result was presented in [13].

5. ASYMPTOTIC BEHAVIOR OF APPROXIMATIONS OF INDIVIDUAL FUNCTIONS IN CLASSES $W^rH_p^\omega$.

Let L_p^r, $r = 0, 1, \ldots$ ($L_p^0 = L_p$), $1 \leqslant p \leqslant \infty$, be the Banach space of 2π-periodic, $r-1$ times continuously differentiable functions f such that $f^{(r-1)}$ is absolutely continuous (for $r \geqslant 1$), the p-th power of $f^{(r)}$ is summable on $(0, 2\pi)$ for $1 \leqslant p < \infty$, and $f^{(r)}$ is essentially bounded for $p = \infty$, with the norm $\|f\|_{pr} = \max\{\|f^{(k)}\|_{L_p(0,2\pi)} : 0 \leqslant k \leqslant r\}$; let C^r, $r = 0, 1, \ldots$ ($C^0 = C$) be the Banach space of 2π-periodic, r times continuously differentiable functions f with the norm $\|f\|_{\infty r}$; let T_n, $n = 0, 1, \ldots$, be the space of trigonometric polynomials of degree at most n; let $W^rH_p^\omega$, $r = 0, 1, \ldots$ ($W^0H_p^\omega = H_p^\omega$), $1 \leqslant p \leqslant \infty$, be the class of functions $f \in L_p^r$, $1 \leqslant p < \infty$, and $f \in C^r$ for $p = \infty$, such that $\omega(f^{(r)}, t)_p \leqslant \omega(t)$, $t \in [0, \pi]$, where $\omega(g, t)_p = \sup\{\|g(\cdot + \delta) - g(\cdot)\|_p : |\delta| \leqslant t\}$ is the modulus of continuity of the function $g \in L_p$, and $\omega(t)$, $t \geqslant 0$, is a given modulus of continuity, i.e. a continuous semiadditive nondecreasing function such that $\omega(0) = 0$.

THEOREM 5.1. [4, 12] *Suppose that* $\omega(t)$ *is a given modulus of continuity satisfying* (12). *Let* $\{ p_n \}_{n=1}^{\infty}$ *be a sequence of continuous seminorms on* L_p^r, $1 \leqslant p < \infty$, *or* C^r ($p = 0$), $r \in \mathbb{N}$.

(a) *If a sequence* $\{ f_n \}_{n=1}^{\infty} \subset W^r H_p^{\omega}$ *satisfies the conditions*

$$\lim_{n\to\infty} \|f_n\|_{pr} = 0 \tag{16}$$

and $p_n(f_n) \neq 0$, $n \in \mathbb{N}$, *then there exists a function* $f_* \in W^r H_p^{\omega}$ *such that* (2) *holds.*

(b) *If* $p_n(f_0) = 0$, *where* $f_0(t) \equiv 1$, *and* $p_n(W^r H_p^{\omega}) \neq 0$, $n \in \mathbb{N}$, *then there exists a function* $f_* \in W^r H_p^{\omega}$ *such that*

$$\limsup_{n\to\infty} \; p_n(f_*) \, / \, p_n(W^r H_p^{\omega}) \;\geqslant\; 1/3.$$

(c) *If* $p_n(f_0) = 0$, *then* $\lim\limits_{n\to\infty} p_n(W^r H_p^{\omega}) = 0$ *if and only if* $\forall f \in W^r H_p^{\omega}$ $\lim\limits_{n\to\infty} p_n(f) = 0$.

It turned out that the condition (16) was satisfied in all instances when p_n was taken as a functional of the best approximation or the remainder of some linear method of approximation and the quantity $p_n(W^r H_p^{\omega})$ was known exactly or asymptotically. It allowed us to obtain not only simpler proofs of some well-known results (see [19]) but the following new assertions.

THEOREM 5.2. [4, 12] *Let* $\omega(t)$ *be an upwards convex modulus of continuity*, $r = 0, 1, \ldots$.

(a) *If* $\omega(t)$ *satisfies* (12), *then there exists a function* $f_* \in W^r H_{\infty}^{\omega}$ *such that*

$$\limsup_{n\to\infty} \; E(f_*, T_n)_{L_1} \, / \, E(W^r H_{\infty}^{\omega}, T_n)_{L_1} \;=\; 1 \, .$$

(b) *If condition* (12) *does not hold, then for each* $f \in W^r H^\omega_\infty$

$$\lim_{n\to\infty} E(f, T_n)_{L_1} / E(W^r H^\omega_\infty, T_n)_{L_1} = 0 .$$

THEOREM 5.3. [3] *Let* $\omega(t)$ *be a given modulus of continuity satisfying* (12), $r = 0, 1, \dots$. *There exists a function* $f_* \in W^r H^\omega_\infty$ *such that*

$$\limsup_{n\to\infty} \|f_* - s_n(f_*)\|_C / e(W^r H^\omega_\infty, s_n)_C = 1 .$$

THEOREM 5.4. [3] *Let* $\omega(t)$ *be an upwards convex modulus of continuity satisfying* (12), $r = 0, 1, \dots$. *There exists a function* $f_* \in W^r H^\omega_1$ *such that*

$$\limsup_{n\to\infty} \|f_* - s_n(f_*)\|_1 / e(W^r H^\omega_1, s_n)_1 = 1 .$$

A linear operator $u_n\colon C \to T_n$ is called a *trigonometric projection* if $u_n(g) = g$ for each $g \in T_n$.

THEOREM 5.5. [4] *Let* $\omega(t) \not\equiv 0$ *be an upwards convex modulus of continuity satisfying* (12), $r = 0, 1, \dots$. *For any arbitrary sequence of trigonometric projections* $u_n\colon C \to T_n$, $n = 0, 1, \dots$, *there exists a function* $f_* \in W^r H^\omega_\infty$ *such that*

$$\limsup_{n\to\infty} \|f_* - u_n(f_*)\|_C / n^{-r} \ln n \int_0^{\pi/2} \omega(2t/n) \sin t \, dt \geqslant \frac{2}{\pi} , \quad (17)$$

and equality holds in (17) *in the case that* $u_n = s_n$.

A similar result was independently obtained by P. O. Runck, J. Szabados and P. Vértesi [21] .

Denote by S_{nm} the space of 2π-periodic splines of order $m \in \mathbb{N}$ of defect 1 with fixed equidistant nodes $x_i = i\pi/n$, $i = 0, 1, \dots, n$, in the interval $[0, 2\pi)$; denote by Q_{nm} the set of all splines of order m of defect 1 with n free nodes in $[0, 2\pi)$.

N. P. Korneichuk [18] posed the problem of finding asymptotic behavior of the best approximation of individual functions by splines with free or fixed nodes. The following four theorems are devoted to the solution of the problem.

THEOREM 5.6. [4] *Let* $\omega(t)$ *be an upwards convex modulus of continuity,* $r = 0, 1, \ldots , m \geq r$.

(a) *If* $\omega(t)$ *satisfies* (12), *then there exists a function* $f_* \in W^r H^\omega_\infty$ *such that*

$$\limsup_{n\to\infty} \; E(f_*, S_{nm})_{L_1} / E(W^r H^\omega_\infty, S_{nm})_{L_1} = 1 .$$

(b) *If condition* (12) *does not hold, then for each* $f \in W^r H^\omega_\infty$

$$\lim_{n\to\infty} \; E(f, S_{nm})_{L_1} / E(W^r H^\omega_\infty, S_{nm})_{L_1} = 0 .$$

Theorem 5.6 is a consequence of Theorem 5.1 . Proofs of the next three theorems are obtained by other means.

THEOREM 5.7. [6] *Let* $\omega(t) \not\equiv 0$ *be an upwards convex modulus of continuity,* $r = 0, 1, \ldots , m \geq r$. *There exists a function* $f_* \in W^r H^\omega_\infty$ *such that*

$$\lim_{n\to\infty} \; E(f_*, S_{nm})_C / E(W^r H^\omega_\infty, S_{nm})_C = 1 .$$

THEOREM 5.8. [6] *Let* $\omega(t)$ *be an upwards convex modulus of continuity satisfying* (12), $r = 0, 1, \ldots , m \geq r$. *There exists a function* $f_* \in W^r H^\omega_\infty$ *such that*

$$\limsup_{n\to\infty} \; E(f_*, Q_{nm})_C / E(W^r H^\omega_\infty, Q_{nm})_C = 1 .$$

THEOREM 5.9. [5] *Suppose that* $\omega(t) = t^\alpha$ *with* $0 < \alpha < 1$. *There exists a function* $f_* \in H^\omega_\infty$ *such that*

$$\liminf_{n\to\infty} \; E(f_*, Q_{n,1})_C / E(H^\omega_\infty, Q_{n,1})_C > 0 .$$

Let $W^rH^\omega[a,b]$, $r = 0, 1, \dots$ ($W^0H^\omega[a,b] = H^\omega[a,b]$), denote the class of functions $f \in C^r[a,b]$ having a given majorant $\omega(t)$ of the modulus of continuity of their r-th derivatives. Obviously, an analog of Theorem 5.1 holds true for the classes $W^rH^\omega[a,b]$.

Let $b : C^r[a,b] \to \mathbb{R}$ be a bounded linear functional on $C^r[a,b]$. A sequence of quadrature formulae

$$q_n(f) = q_n(f;\ X_n;\ P_n) = \sum_{k=0}^{n} \sum_{\nu=0}^{\rho} p_{kn}^{(\nu)} f^{(\nu)}(x_{kn}) \ , \quad n = 0,\ 1, \dots \quad (18)$$

is defined by sequences of nodes $X_n = \{ a \leqslant x_{1,n} < x_{2,n} < \dots < x_{n,n} \leqslant b \}$ and coefficients $P_n = \{ p_{kn}^{(\nu)} : k = 0, \dots, n ,\ \nu = 0, \dots, \rho \}$, $0 \leqslant \rho \leqslant r$.

Denote

$$R_b(\mathfrak{M},\ q_n) := \sup \{ |b(f) - q_n(f)| : f \in \mathfrak{M} \} \ , \quad \mathfrak{M} \subset C^r \ ,$$

$$R_b(\mathfrak{M},\ X_n,\ \rho) := \inf_{P_n} R_b(\mathfrak{M},\ q_n) \ ,$$

$$d(X_n) := \max \{ x_{o,n} - a \ , \ b - x_{n,n} \ , \ x_{i,n} - x_{i-1,n} : i = 1, \dots, n \} \ .$$

A sequence of quadrature formulae q_n is called to be asymptotically optimal on the class $\mathfrak{M}$ with respect to coefficients if

$$\lim_{n\to\infty} R_b(\mathfrak{M},\ q_n) \ / \ R_b(\mathfrak{M},\ X_n,\ \rho) = 1 \ .$$

Theorem 5.10. [2] *Suppose that* $\omega(t)$ *is a given modulus of continuity satisfying* (12), $r = 0, 1, \dots,$ b *is a bounded linear functional on* $C^r[a,b]$, $\{ q_n \}_{n=0}^{\infty}$ *is a sequence of quadrature formulae of type* (18). *If* $\lim_{n\to\infty} d(X_n) = 0$ *and*

$R_b(W^rH^\omega[a,b],\ X_n,\ \rho) \neq 0$, $n = 0,\ 1, \ldots,$ *then there exists a function* $f_* \in W^rH^\omega[a,b]$ *such that*

$$\limsup_{n\to\infty}\ |b(f_*) - q_n(f_*)| \,/\, R_b(W^rH^\omega[a,b],\ X_n,\ \rho)\ \geqslant\ 1\ , \quad (19)$$

and equality holds in (19) *in the case that the sequence* $\{\, q_n \,\}_{n=1}^{\infty}$ *is asymptotically optimal on the class* $W^rH^\omega[a,b]$ *with respect to coefficients.*

An infinite triangular matrix $M = (x_{kn})$, $-1 \leqslant x_{0,n} < x_{1,n} < \ldots < x_{n,n} \leqslant 1$, $n = 0,\ 1, \ldots$ defines a Lagrange interpolation process

$$L_n(M,\ f,\ t)\ =\ \sum_{k=0}^{n} f(x_{kn})\ l_{kn}(M,\ t)\ ,$$

$$l_{kn}(M,\ t)\ :=\ \prod_{\substack{i=0\\ i\neq k}}^{n} (t - x_{in}) \,/\, (x_{kn} - x_{in})\ .$$

An interpolation process $L_n(M)$ is said to be *convergent at a point* $t \in [-1,1]$ (*uniformly convergent on* $[-1,1]$) *in the class* $\mathfrak{M} \subset C[-1,1]$ if for any $f \in \mathfrak{M}$ $\lim\limits_{n\to\infty} |f(t) - L_n(M,\ f,\ t)| = 0$ ($\lim\limits_{n\to\infty} \|f - L_n(M,\ f)\|_{C[-1,1]} = 0$) .

The following criterion of convergence of interpolation processes in classes $W^rH^\omega[-1,1]$ is a consequence of Corollary 2.5.

Theorem 5.11. [7] *Let* $\omega(t)$ *be a given modulus of continuity satisfying* (12), $r = 0,\ 1, \ldots$.

(a) $L_n(M)$ *is convergent at the point* $t \in [-1,1]$ *in the class* $W^rH^\omega[-1,1]$ *if and only if*

$$\lim_{n\to\infty}\ \sup\ \{\ |f(t) - L_n(M,\ f,\ t)|\ :\ f \in W^rH^\omega[-1,1]\ \}\ =\ 0\ .$$

(b) $L_n(M)$ *is uniformly convergent on* $[-1,1]$ *in the class* $W^rH^\omega[-1,1]$ *if and only if*

$$\lim_{n\to\infty} \sup \{ \|f - L_n(M, f)\|_{C[-1,1]} : f \in W^rH^\omega[-1,1] \} = 0 .$$

Denote

$$\delta_n(M) := \min_{1\le k\le n} |\arccos x_{kn} - \arccos x_{k-1,n}| , \quad n \in \mathbb{N} .$$

By applying Theorem 5.11 we obtained the following extension of a Vértesi's result [22].

THEOREM 5.12. [7] *Let* $\omega(t)$ *be a given modulus of continuity satisfying* (12), $r = 0, 1, \ldots$. *Suppose that*

$$\liminf_{n\to\infty} n\,\delta_n(M) > 0 .$$

Then

(a) $L_n(M)$ *is convergent at the point* $t \in [-1,1]$ *in the class* $W^rH^\omega[-1,1]$ *if and only if*

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{(1-x_{kn}^2)^{r/2}}{n^r}\, \omega\left(\frac{(1-x_{kn}^2)^{1/2}}{n} \right) |l_{kn}(M, t)| = 0 .$$

(b) $L_n(M)$ *is uniformly convergent on* $[-1,1]$ *in the class* $W^rH^\omega[-1,1]$ *if and only if*

$$\lim_{n\to\infty} \max_{-1\le t\le 1} \sum_{k=0}^{n} \frac{(1-x_{kn}^2)^{r/2}}{n^r}\, \omega\left(\frac{(1-x_{kn}^2)^{1/2}}{n} \right) |l_{kn}(M, t)| = 0 .$$

P. Vértesi [22] presented such a theorem in the case of $r = 0$ for matrices M consisting of zeros of Jacobi polynomials.

6. Accuracy of Seminorm Inequalities

Let X be a Banach space, $\{ p_n \}_{n=1}^{\infty}$ and $\{ \omega_n \}_{n=1}^{\infty}$ be two sequences of continuous seminorms on X. Let us consider the quantity $\text{æ}_n = \text{æ}_n(X, p, \omega) := \sup \{ p_n(f) / \omega_n(f) : f \in X , \omega_n(f) \neq 0 \}$, $n \in \mathbb{N}$. Under the assumption that $0 < \text{æ}_n < \infty$, $n \in \mathbb{N}$, a sequence of elements $f_n \in X$ is said to be (asymptotically) *extremal* for $\text{æ}_n(X, p, \omega)$ if

$$\lim_{n\to\infty} p_n(f_n) / \text{æ}_n \omega_n(f_n) = 1 .$$

The following theorem that was proved by using Theorem 2.1 investigates asymptotic behavior of the ratio

$$p_n(f) / \omega_n(f) , \quad n \to \infty , \tag{20}$$

for individual elements $f \in X$ not depending on n.

Theorem 6.1. [11] *Suppose that* $0 < \text{æ}_n(X, p, \omega) < \infty$, $n \in \mathbb{N}$, *and there exists a positive sequence of real numbers* $\alpha = \{ \alpha_n \}_{n=1}^{\infty}$ *such that the following two conditions hold.*

A. The set $X(\omega, \alpha) := \{ f \in X : \lim\limits_{n\to\infty} \omega_n(f) / \alpha_n = 0 \}$ *is everywhere dense in* X.

B. There exists an extremal sequence $\{ f_n \}_{n=1}^{\infty}$ *for* $\text{æ}_n(X, p, \omega)$ *such that*

$$\lim_{n\to\infty} \alpha_n \|f_n\|_X / \omega_n(f_n) = 0 .$$

Then there exists an element $f_* \in X$ *such that*

$$\limsup_{n\to\infty} p_n(f_*) / \text{æ}_n \omega_n(f_*) = 1 .$$

There are many consequences of this result. First of all, it can be applied to the study of the accuracy of Jackson type inequalities. Let Y^r be one of the spaces L_q^r, $1 \leqslant q \leqslant \infty$, C^r, $r = 0, 1, \ldots$, $\omega(g, t)_Y$, $g \in Y$, $t \in [0, \pi]$, be the Y-modulus of continuity, i.e. $\omega(g, t)_Y = \omega(g, t)_q$ if $Y = L_q$, $1 \leqslant q < \infty$, and $\omega(g, t)_Y = \omega(g, t)_\infty$ if $Y = C$. For any positive number sequence $\gamma_n \to 0$ $(n \to \infty)$ we consider the sequence of continuous seminorms $\omega_n(f) = \omega(f^{(r)}, \gamma_n)_Y$ on Y^r.

THEOREM 6.2. [11] *Let Y^r be L_q^r, $1 \leqslant q < \infty$ or C^r, $r = 0, 1, \ldots$, $\{ p_n \}_{n=1}^{\infty}$ be an arbitrary sequence of continuous seminorms on Y^r, $\omega_n(f) = \omega(f^{(r)}, \gamma_n)_Y$, $\gamma_n > 0$, $\gamma_n \to 0$ $(n \to \infty)$. Suppose that $0 < æ_n(Y^r, p, \omega) < \infty$ and there exists an extremal sequence $\{ f_n \}_{n=1}^{\infty}$ for $æ_n(Y^r, p, \omega)$ such that*

$$\lim_{n \to \infty} \gamma_n \| f_n^{(r)} \|_Y / \omega(f_n^{(r)}, \gamma_n)_Y = 0 .$$

Then there exists a function $f_* \in Y^r$ such that

$$\limsup_{n \to \infty} p_n(f_*) / æ_n \, \omega(f_*^{(r)}, \gamma_n)_Y = 1 .$$

According to a Korneichuk's theorem, for any $f \in C$

$$E(f, T_n)_C < \omega(f, \pi/n)_C \tag{22}$$

while for $\varepsilon > 0$ for each $n \in \mathbb{N}$ there exists a function $f_{n,\varepsilon} \in C$ such that

$$E(f_{n,\varepsilon}, T_n)_C > \left(1 - \frac{1}{2n} - \varepsilon \right) \omega(f_{n,\varepsilon}, \pi/n)_C .$$

By applying Theorem 6.2 we find out that inequality (22) can not be improved even for individual functions.

COROLLARY 6.3. [8] *There exists a function* $f_* \in C$ *such that*

$$\limsup_{n\to\infty} E(f_*, T_n)_C \,/\, \omega(f_*, \pi/n)_C = 1 . \qquad (23)$$

In a similar manner we deduce the accuracy of some other inequalities.

COROLLARY 6.4. [11] *There exists a function* $f_* \in L_2$ *such that*

$$\limsup_{n\to\infty} E(f_*, T_n)_2 \,/\, \omega(f_*, \pi/n)_2 = 1/\sqrt{2} .$$

COROLLARY 6.5. [11] *For each odd* $r = 1, 3, 5, \ldots$ *there exists a function* $f_* \in L_1^r$ *such that*

$$\limsup_{n\to\infty} n^r E(f_*, T_n)_1 \,/\, \omega(f_*^{(r)}, \pi/n)_1 = K_r / 2 ,$$

where K_r *is the Favard constant.*

COROLLARY 6.6. [11] *There exists a function* $f_* \in C$ *such that*

$$\limsup_{n\to\infty} \|f_* - s_n(f_*)\|_C \,/\, \omega\left(f_*, \tfrac{4\pi}{3(2n+1)}\right)_C \ln n = 2/\pi^2 .$$

Let $t_n(f)$ be the trigonometric polynomial of degree at most $n-1$ that interpolates $f \in C$ at points $(2k-1)\pi/(2n-1)$, $k = 1, \ldots, 2n-1$, $\|t_n\|$ be the Lebesgue constant of the corresponding interpolation process.

COROLLARY 6.7. [11] *There exists a function* $f_* \in C$ *such that*

$$\limsup_{n\to\infty} \|f_* - t_n(f_*)\|_C \,/\, \|t_n\| \,\omega\left(f_*, \tfrac{2\pi}{(2n-1)}\right)_C = 1/2 .$$

The last Corollary improves a result of O. Kis [17].

Returning to the inequality (22), we should point out the following result.

Theorem 6.8. [8] *For any function* $f \in C$

$$\liminf_{n\to\infty} E(f, T_n)_C / \omega(f, \pi/n)_C \leqslant 1/2$$

and there exists a function $f_* \in C$ *such that*

$$\liminf_{n\to\infty} E(f_*, T_n)_C / \omega(f_*, \pi/n)_C = 1/2 .$$

Comparison of Theorem 6.8 and Corollary 6.3 shows that there exists a gap between upper and lower limits in the case. The situation reverses if we consider Jackson inequality for differentiable functions $f \in C^r$, where r is odd,

$$E(f, T_n)_C \leqslant (\mathcal{K}_r/2n^r)\, \omega(f^{(r)}, \pi/n)_C .$$

Theorem 6.9. [8] *For each odd* $r = 1, 3, 5, \dots$ *there exists a function* $f_* \in C^r$ *such that*

$$\lim_{n\to\infty} n^r E(f_*, T_n)_C / \omega(f_*^{(r)}, \pi/n)_C = \mathcal{K}_r / 2 .$$

Let us consider the Lebesgue's inequality

$$\|f - s_n(f)\|_C \leqslant \left[4/\pi^2 \ln n + O(1) \right] E(f, T_n)_C , \quad f \in C . \qquad (24)$$

As an answer to the question whether it is accurate for individual functions, we have obtained the following general theorem which is a consequence of Theorem 6.1.

Let $\{ F_n \}_{n=1}^{\infty}$ be a sequence of subspaces in a Banach space X with $F_n \subset F_{n+1}$, $n \in \mathbb{N}$ and $\bigcup_{n\in\mathbb{N}} F_n$ is everywhere dense in X but not equal to X .

Theorem 6.10. [11] *Let* $\{ p_n \}_{n=1}^{\infty}$ *be an arbitrary sequence of continuous seminorms on* X ,

$$\varkappa_n = \sup \{ p_n(f) / E(f, F_n)_X : f \in X \setminus F_n \} .$$

If $0 < \varkappa_n < \infty$, $n \in \mathbb{N}$, then there exists an element $f_* \in X$ *such that*

$$\limsup_{n\to\infty} \ p_n(f_*) \ / \ æ_n \ E(f_*, F_n)_X = 1 .$$

COROLLARY 6.11. [11] *There exists a function* $f_* \in C$ *such that*

$$\limsup_{n\to\infty} \ \|f_* - s_n(f_*)\|_C \ / \ E(f_*, T_n)_C \ln n = 4/\pi^2 .$$

REFERENCES

1. P.L.BUTZER AND W.DICKMEIS, On the sharpness of non-optimal approximation for semigroup operators, *Manuscripta Math.* **51** (1985), p.243-251.
2. O.V.DAVYDOV, On the asymptotic behavior of remainder of quadrature formulae for individual functions of classes W^rH_ω, Deponirovannaya rukopis no.4437-V87, VINITI, 1987 (in Russian).
3. O.V.DAVYDOV, On the approximation of individual functions by Fourier sums and their means, in "Questions of Optimal Approximation of Functions and Summation of Series" (V.P.Motornyi, Ed.), p.20-23, Dnepropetrovsk. Gos. Univ., Dnepropetrovsk, 1988 (in Russian).
4. O.V.DAVYDOV, Asymptotic behavior of best approximations of individual functions of classes W^rH_ω in the metric L_p, *Dokl. Akad. Nauk SSSR* **306** (1989), no.4, p.777-781 (in Russian); English translation in *Soviet Mat. Dokl.* **39** (1989), no.3, p.541-544; MR 90m: 41038.
5. O.V.DAVYDOV, Approximation of individual functions of the class H^α by broken lines with free nodes, in "Theory of Approximation of Functions and Summation of Series" (V.P.Motornyi, Ed.), p.17-21, Dnepropetrovsk. Gos. Univ., Dnepropetrovsk, 1989 (in Russian).

6. O.V.DAVYDOV, Asymptotic behavior of the best uniform approximations of individual functions by splines, *Ukrainian Mat. Zh.* **42** (1990), no.1, p.59-64 (in Russian); English translation in *Ukrainian Math. J.* **42** (1990), no.1, p.50-54; MR 92c: 42001.

7. O.V.DAVYDOV, Convergence of Lagrange interpolation processes in classes of differentiable functions, *Izv. Vyssh. Uchebn. Zaved. Mat.* 1990, no.4, p.74-77 (in Russian); English translation in *Soviet Math. (Iz. VUZ)* **34** (1990), no.4, p.89-92.

8. O.V.DAVYDOV, Accuracy of Jackson inequality for individual functions, *Ukrain. Mat. Zh.* **42** (1990), no.11, p.1469-1475 (in Russian); English translation in *Ukrainian Math. J.* **42** (1990), no.11, p.1311-1316; MR 92c:42001.

9. O.V.DAVYDOV, Condensation of values of seminorm sequences on the sets defined by means of majorants of the best approximations, in "Approximation of Functions by Polynomials and Splines and Summation of Series" (V.P.Motornyi, Ed.), p.18-24, Dnepropetrovsk. Gos. Univ., Dnepropetrovsk, 1990 (in Russian).

10. O.V.DAVYDOV, On the approximation of functions of class $C(\varepsilon)$ by subsequences of de la Vallée-Poussin sums, *Mat. Zametki* **49** (1991), no.6, p.33-41 (in Russian); English translation in *Math. Notes* **49** (1991), p.577-583.

11. O.V.DAVYDOV, On the accuracy of inequalities of Jackson and Lebesgue type, in "Approximation of Functions and Summation of Series" (V.P.Motornyi, Ed.), p.19-28, Dnepropetrovsk. Gos. Univ., Dnepropetrovsk, 1991 (in Russian).

12. O.V.DAVYDOV, On the condensation of values for sequences of seminorms, *Ukrain. Mat. Zh.* **44** (1992), no.4, p.467-474 (in Russian); English translation in *Ukrainian Math. J.* **44** (1992), no.4, p.409-416.

13. O.V.DAVYDOV, On the approximation of individual functions by rectangular Fourier sums, *Dokl. Akad. Nauk (Russia)* **327** (1992), no.3, p.295-298 (in Russian).

14. O.V.Davydov, V.A.Kofanov and S.A.Pichugov, On the residuality of sets of extremals for approximation of functional classes, in "Optimization of Methods of Approximation" (N.P.Korneichuk, Ed.), p.42-50, Akad. Nauk Ukrain.SSR, Inst. Mat., Kiev, 1992 (in Russian).

15. O.V.Davydov, Some condensation theorems for semigroup operators, *Manuscripta Math.* **79** (1993), p.435-446.

16. W.Dickmeis, R.J.Nessel and E.van Wickeren, Quantitative extension of the uniform boundedness principle, *Jahresber. Deutsch. Math. Ver.* **89** (1987), no.3, p.105-134.

17. O.Kis, Remarks on the estimation of error of interpolation, *Studia Scient. Math. Hung.* **4** (1969), no.1-4, p.279-289 (in Russian).

18. N.P.Korneichuk, On some problems of polynomial and spline-approximation, in "The Constructive Theory of Functions. Proc. Intl Conf. (Varna, May 27 - June 2, 1984)", p.49-60, Bulgarian Academy of Sciences, 1984 (in Russian).

19. N.P.Korneichuk, S.M.Nikol'skii and the development of the study of the theory of approximation of functions in the USSR, *Uspekhi Mat. Nauk* **40** (1985), no.5, p.71-131 (in Russian).

20. K.I.Oskolkov, An inequality of Lebesgue in the uniform metric and on a set of complete measure, *Mat. Zametki* **18** (1975), no.4, 515-526 (in Russian).

21. P.O.Runck, J.Szabados and P.Vértesi, On the convergence of the differentiated trigonometric projection operators, *Acta Sci. Math.* **53** (1989), 287-293.

22. P.Vértesi, Estimations for some interpolatory processes, *Acta Math. Acad. Scient. Hung.* **27** (1976), no.1-2, p.109-119.

23. E.van Wickeren, A Baire approach to quantitative resonance principles, *Numer. Anal. Optimiz.* **3** (1987), no.1&2, p.147-180.