Approximation by sums of piecewise linear polynomials

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Abstract

We present two partitioning algorithms that allow a sum of piecewise linear polynomials over a number of overlaying convex partitions of the unit cube Ω in $\mathbb{R}^d$ to approximate a function $f \in W_p^3(\Omega)$ with the order $N^{-6/(2d+1)}$ in L_p-norm, where N is the total number of cells of all partitions, which makes a marked improvement over the $N^{-2/d}$ order achievable on a single convex partition. The gradient of f is approximated with the order $N^{-3/(2d+1)}$. The first algorithm creates d convex partitions and relies on the knowledge of the eigenvectors of the average Hessians of f over the cells of an auxiliary uniform partition, whereas the second algorithm with $\binom{d+1}{2}$ convex partitions is independent of f. In addition, we also give an f-independent partitioning algorithm for a sum of d piecewise constants that achieves the approximation order $N^{-2/(d+1)}$.

1 Introduction

Let $\Omega = (0,1)^d$, $d \geq 2$. A finite set Δ of subdomains ω of Ω (called *cells*) is said to be a *partition* of Ω if $\omega \cap \omega' = \emptyset$ when $\omega \neq \omega'$, and $\sum_{\omega \in \Delta} |\omega| = |\Omega|$, where $|\omega|$ denotes the Lebesgue measure (d-dimensional volume) of ω. A partition is *convex* if each cell ω is a convex domain. The cardinality of a finite set D is denoted $|D|$, so that $|\Delta|$ stands for the number of cells ω in the partition Δ.

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Given a partition Δ, the linear space of piecewise polynomials of order k with respect to it is defined by

$$S_k(\Delta) = \Big\{ \sum_{\omega\in\Delta} q_\omega \chi_\omega : \; q_\omega \in \Pi_k^d \Big\}, \qquad \chi_\omega(x) := \begin{cases} 1, & \text{if } x \in \omega, \\ 0, & \text{otherwise}, \end{cases}$$

where Π_k^d, $k \geq 1$, is the space of polynomials of total degree $< k$ in d variables. The error of the best L_p-approximation of a function $f \in L_p(\Omega)$ from $S_k(\Delta)$,

$$E_k(f,\Delta)_p := \inf_{s\in S_k(\Delta)} \|f-s\|_p, \qquad 1 \leq p \leq \infty,$$

can be computed if the errors $E_k(f)_{L_p(\omega)} := \inf_{q\in\Pi_k^d} \|f-q\|_{L_p(\omega)}$ of the best polynomial approximations of f on all $\omega \in \Delta$ are known. Indeed,

$$E_k(f,\Delta)_p = \begin{cases} \Big(\sum_{\omega\in\Delta} E_k(f)^p_{L_p(\omega)} \Big)^{1/p} & \text{if } \; p < \infty, \\ \max_{\omega\in\Delta} E_k(f)_{L_\infty(\omega)} & \text{if } \; p = \infty. \end{cases} \tag{1}$$

For a system $\mathcal{P} = \{\Delta^{(1)}, \ldots, \Delta^{(n)}\}$ of several overlaying partitions of Ω, we consider the *space of sums of piecewise polynomials*

$$S_k(\mathcal{P}) = \Big\{ \sum_{\nu=1}^n \sum_{\omega\in\Delta^{(\nu)}} q_{\nu,\omega}\chi_\omega : \; q_{\nu,\omega} \in \Pi_k^d \Big\}.$$

Thus, a function s in $S_k(\mathcal{P})$ is the sum of n piecewise polynomials $s = \sum_{\nu=1}^n s_\nu$ with $s_\nu \in S_k\big(\Delta^{(\nu)}\big)$, $\nu = 1,\ldots,n$. We set $|\mathcal{P}| := \sum_{\nu=1}^n |\Delta^{(\nu)}|$ and denote the best approximation error from $S_k(\mathcal{P})$ by

$$E_k(f,\mathcal{P})_p := \inf_{s\in S_k(\mathcal{P})} \|f-s\|_p, \qquad 1 \leq p \leq \infty.$$

Given a function f, we consider piecewise polynomial approximations of f on suitably designed partitions. Standard uniform type partitions deliver piecewise polynomial approximations with the order

$$E_k(f,\Delta)_p = \mathcal{O}(|\Delta|^{-k/d}), \qquad |\Delta| \to \infty, \tag{2}$$

if f belongs to the Sobolev space $W_p^k(\Omega)$, as follows from the Bramble-Hilbert lemma, see for example [3].

It is shown in [1, Theorem 2] that the approximation order of piecewise constants $E_1(f,\Delta)_\infty = \mathcal{O}(|\Delta|^{-1/d})$ cannot be improved even assuming infinite differentiability of f if the partitions are isotropic. One thus has to use anisotropic partitions if smoothness should pay off in convergence rate.

A simple algorithm suggested in [1, 2] (see Algorithm 1 and Theorem 1 below) delivers an improved approximation order $E_1(f,\Delta)_p = \mathcal{O}(|\Delta|^{-2/(d+1)})$ of piecewise constants on suitable anisotropic convex partitions if $f \in W_p^2(\Omega)$. Here, the unit cube is first subdivided uniformly into m^d subcubes (macro-cells), each of which is then splitted anisotropically into m slices (micro-cells). (Note that in the case $d = 2$ the order $E_1(f,\Delta)_p = \mathcal{O}(|\Delta|^{-2/3})$ has been obtained earlier in [4] by a different method.) Moreover, [2, Theorem 2] shows that $|\Delta|^{-2/(d+1)}$ is the saturation order of piecewise constant approximation on convex partitions in the sense that it cannot be further improved for any $f \in C^2(\Omega)$ whose Hessian is positive definite at some point. Nevertheless, [2, Theorem 3] suggests that this phenomenon is restricted to piecewise constants, as the saturation order of piecewise linear approximations on convex partitions is $|\Delta|^{-2/d}$, that is the same as on the isotropic partitions.

In this paper we show that for $k = 2$ the approximation order $E_2(f,\Delta)_p = \mathcal{O}(|\Delta|^{-2/d})$ in (2) can be improved to $E_2(f,\mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-6/(2d+1)})$ if $f \in W_p^3(\Omega)$ by using a sum of piecewise linear polynomials with respect to a system $\mathcal{P}$ of d convex polyhedral partitions of Ω (Algorithm 3 and Theorem 3). Moreover, the approximation of the gradient of f improves to $\mathcal{O}(|\mathcal{P}|^{-3/(2d+1)})$ from the standard estimate $\mathcal{O}(|\Delta|^{-1/d})$ for piecewise linear polynomials on a single partition.

In addition, we show that the sums of d piecewise constants on suitable *fixed, f-independent partitions* can be used to obtain the same approximation order $E_1(f,\mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-2/(d+1)})$ (Algorithm 2 and Theorem 2), whereas Algorithm 1 relies on the knowledge of the average gradients of f on the macro-cells. Similarly, the sums of $\binom{d+1}{2}$ piecewise linear polynomials on f-independent partitions can be used to obtain the approximation order $E_2(f,\mathcal{P})_p = \mathcal{O}(|\mathcal{P}|^{-6/(2d+1)})$ (Algorithm 4 and Theorem 4), whereas Algorithm 3 employs the average Hessians of f on the macro-cells.

The results presented here are based on Chapter 4 of the thesis of the second named author [5].

The paper is organised as follows. Section 2 is devoted to the piecewise constant approximation, where after recalling the algorithm suggested in [1, 2] we present our new result for the sums of piecewise constants with fixed splitting directions of the macro-cells, whereas in Sections 3 and 4 we describe the two algorithms for the sums of piecewise linear polynomials.

In what follows we will use the following version of the Sobolev seminorm

$$|f|_{W_p^n(\omega)} := \sum_{|\alpha|=n} \Big\|\frac{\partial^n f}{\partial x^\alpha}\Big\|_{L_p(\omega)}, \quad |\alpha| := \alpha_1 + \cdots + \alpha_d \ \text{ for } \ \alpha \in \mathbb{Z}_+^d,$$

and recall that if $\omega \subset \mathbb{R}^d$ is a bounded convex domain and $f_{|\omega} \in W_p^k(\omega)$, then there exists a polynomial $q \in \Pi_k^d$ such that [3]

$$|f-q|_{W_p^r(\omega)} \le \rho_{d,k} \operatorname{diam}^{k-r}(\omega)|f|_{W_p^k(\omega)}, \quad r=0,\dots,k, \tag{3}$$

where $\rho_{d,k}$ denotes a positive constant depending only on d and k [3]. In view of Lemma 1, (3) implies in particular the Poincaré inequality

$$\|f-f_\omega\|_{L_p(\omega)} \le \rho_d \operatorname{diam}(\omega)\|\nabla f\|_{L_p(\omega)}, \qquad f \in W_p^1(\omega), \tag{4}$$

with a constant ρ_d depending only on d, where $f_\omega := |\omega|^{-1}\int_\omega f(x)\,dx$ and

$$\|\nabla f\|_{L_p(\omega)} := \Big\|\Big(\sum_{k=1}^d |D_{x_k} f|^2\Big)^{1/2}\Big\|_{L_p(\omega)}, \qquad D_{x_k} f := \frac{\partial f}{\partial x_k}.$$

Note that $\|f_\omega - c\|_{L_p(\omega)} \le \|f-c\|_{L_p(\omega)}$ for any constant c, and hence $\|f - f_\omega\|_{L_p(\omega)} \le 2E_1(f)_{L_p(\omega)}$. We prefer to use (4) rather than (3) when $k=1$ because explicit values or estimates of the optimal constant in (4) are known for $p = 1, 2, \infty$, see a discussion and references in [1, Section 2].

Lemma 1. *For any* $1 \le p \le \infty$,

$$\|\nabla f\|_{L_p(\omega)} \le |f|_{W_p^1(\omega)} \le d^{\max\{\frac12, 1-\frac1p\}}\|\nabla f\|_{L_p(\omega)}. \tag{5}$$

Proof. By the inequality between discrete 2- and 1-norms, and triangle inequality, we have

$$\|\nabla f\|_{L_p(\omega)} \le \Big(\int_\omega \Big(\sum_{k=1}^d |D_{x_k} f(x)|\Big)^p dx\Big)^{1/p} \le \sum_{k=1}^d \Big(\int_\omega |D_{x_k} f(x)|^p dx\Big)^{1/p},$$

which shows the first inequality in (5). The second one is obtained as follows. By the inequality between arithmetic and p-power means,

$$|f|_{W_p^1(\omega)} \le d^{1-\frac1p}\Big(\sum_{k=1}^d \int_\omega |D_{x_k} f(x)|^p dx\Big)^{1/p},$$

which completes the proof if $p=2$. If $p>2$, then $|f|_{W_p^1(\omega)} \le d^{1-\frac1p}\|\nabla f\|_{L_p(\omega)}$ follows by the inequality between p- and 2-norms. If $p<2$, then the inequality between p- and 2-means leads to $|f|_{W_p^1(\omega)} \le d^{\frac12}\|\nabla f\|_{L_p(\omega)}$. □

2 Sums of piecewise constants

The following algorithm for piecewise constant approximation with optimal approximation order $|\Delta|^{-2/(d+1)}$ on convex polyhedral partitions has been suggested in [1, 2].

Algorithm 1 ([2]). *Assume $f \in W_1^1(\Omega)$, $\Omega = (0,1)^d$. Split Ω into $N_1 = m^d$ cubes $\omega_1, \dots, \omega_{N_1}$ of edge length $1/m$, with $m \in \mathbb{Z}_+$. Then split each ω_i into N_2 slices ω_{ij}, $j = 1, \dots, N_2$, by equidistant hyperplanes orthogonal to the average gradient $g_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f(x)\, dx$ on ω_i. Set $\Delta = \{\omega_{ij} : i = 1, \dots, N_1,\ j = 1, \dots, N_2\}$. Clearly, $|\Delta| = N_1 N_2$ and each ω_{ij} is a convex polyhedron with at most $2(d+1)$ facets.*

Theorem 1 ([2]). *Assume that $f \in W_p^2(\Omega)$, $\Omega = (0,1)^d$, for some $1 \le p \le \infty$. For any $m = 1, 2, \dots$, generate the partition Δ_m by using Algorithm 1 with $N_1 = m^d$ and $N_2 = m$. Then*

$$E_1(f, \Delta_m)_p \le C|\Delta_m|^{-2/(d+1)}(|f|_{W_p^1(\Omega)} + |f|_{W_p^2(\Omega)}), \tag{6}$$

where C is a constant depending only on d.

The new algorithm will involve a system of d convex polyhedral partitions *independent* of f.

Algorithm 2. *Split $\Omega = (0,1)^d$ into $N_1 = m^d$, $m \in \mathbb{Z}_+$, cubes $\omega_1, \dots, \omega_{N_1}$ of edge length $1/m$, whose edges are parallel to the coordinate axes. For each $\nu = 1, \dots, d$, define $\Delta^{(\nu)}$ by splitting each ω_i into N_2 slices $\omega_{ij}^{(\nu)}$, $j = 1, \dots, N_2$, by equidistant hyperplanes orthogonal to the x_ν-axis. Set $\mathcal{P} = \{\Delta^{(1)}, \dots, \Delta^{(d)}\}$. Then $|\Delta^{(\nu)}| = N_1 N_2$ for all $\nu = 1, \dots, d$ and $|\mathcal{P}| = dN_1 N_2$.*

Partitions $\Delta^{(1)}, \Delta^{(2)}$ in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Fig. 1. Note that each $\omega_{ij}^{(\nu)}$ is a d-dimensional box with its ν-th dimension $\frac{1}{mN_2}$ and all other dimensions $\frac{1}{m}$.

Theorem 2. *Assume that $f \in W_p^2(\Omega)$, $\Omega = (0,1)^d$, for some $1 \le p \le \infty$. For any $m = 1, 2, \dots$, generate the system of partitions $\mathcal{P}_m$ by using Algorithm 2 with $N_1 = m^d$ and $N_2 = m$. Then*

$$E_1(f, \mathcal{P}_m)_p \le C|\mathcal{P}_m|^{-2/(d+1)}(|f|_{W_p^1(\Omega)} + |f|_{W_p^2(\Omega)}), \tag{7}$$

where C is a constant depending only on d.

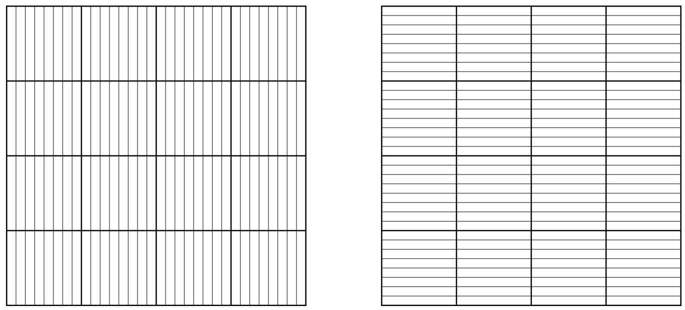

Figure 1: Partitions $\Delta^{(1)}, \Delta^{(2)}$ for piecewise constant approximation ($d = 2$, $N_2 = m = 4$).

Proof. For the sake of brevity we assume that $p < \infty$. The proof for $p = \infty$ is the same except that at the appropriate places integrals have to be replaced by the L_∞-norm.

We first introduce an auxiliary piecewise linear approximation of f. For each $i = 1, \dots, N_1$, let

$$\ell_i = c_i + \sum_{\nu=1}^{d} \ell_{i,\nu},$$

where

$$c_i = |\omega_i|^{-1} \int_{\omega_i} f(x)\, dx, \qquad \ell_{i,\nu}(x) = a_{i,\nu}(x_\nu - x_{i,\nu}),$$

with

$$a_{i,\nu} := |\omega_i|^{-1} \int_{\omega_i} D_{x_\nu} f(x)\, dx, \quad \nu = 1, \dots, d,$$

and $(x_{i,1}, \dots, x_{i,d})$ denotes the barycenter of ω_i. We set

$$\ell := \sum_{i=1}^{N_1} \ell_i \chi_{\omega_i}.$$

Since

$$\int_{\omega_i} \ell_{i,\nu}(x)\, dx = 0, \quad \nu = 1, \dots, d,$$

and $\operatorname{diam}(\omega_i) \le \frac{\sqrt{d}}{m}$, we deduce by the Poincaré inequality and (5),

$$\begin{aligned} \Big\| \Big(f - \sum_{\nu=1}^{d} \ell_{i,\nu} \Big) - c_i \Big\|_{L_p(\omega_i)} &\le \rho_d \operatorname{diam}(\omega_i) \Big\| \nabla f - \sum_{\nu=1}^{d} \nabla \ell_{i,\nu} \Big\|_{L_p(\omega_i)} \\ &\le \frac{\sqrt{d}\rho_d}{m} \sum_{\nu=1}^{d} \| D_{x_\nu} f - a_{i,\nu} \|_{L_p(\omega_i)}. \end{aligned}$$

The Poincaré inequality and (5) also imply

$$\begin{aligned}\|D_{x_\nu} f - a_{i,\nu}\|_{L_p(\omega_i)} &\le \rho_d \operatorname{diam}(\omega_i) \|\nabla(D_{x_\nu} f)\|_{L_p(\omega_i)} \\ &\le \frac{\sqrt{d}\rho_d}{m} \sum_{\mu=1}^{d} \|D_{x_\mu x_\nu} f\|_{L_p(\omega_i)},\end{aligned}$$

where we set $D_{x_\mu x_\nu} f := D_{x_\mu} D_{x_\nu} f$. It follows that

$$\|f - \ell_i\|_{L_p(\omega_i)} \le \frac{d\rho_d^2}{m^2} \sum_{\nu,\mu=1}^{d} \|D_{x_\mu x_\nu} f\|_{L_p(\omega_i)} = \frac{2d\rho_d^2}{m^2} |f|_{W_p^2(\omega_i)}.$$

Hence,

$$\|f - \ell\|_p = \Big(\sum_{i=1}^{N_1} \|f - \ell_i\|_{L_p(\omega_i)}^p \Big)^{\frac{1}{p}} \le \frac{2d\rho_d^2}{m^2} |f|_{W_p^2(\Omega)}. \tag{8}$$

For each i, ν, let $[x_j^0, x_j^1]$ be the projection of the ν-th edge of $\omega_{ij}^{(\nu)}$ on the x_ν-axis, $j = 1, \ldots, N_2$. We now replace $\ell_{i,\nu}$ by the piecewise constant function

$$s_{i,\nu} = \sum_{j=1}^{N_2} b_j a_{i,\nu} \chi_{\omega_{ij}^{(\nu)}},$$

where $b_j = x_j^0 - x_{i,\nu}$, $j = 1, \ldots, N_2$. Since $x_j^1 - x_j^0 = \frac{1}{mN_2}$, we obtain

$$\begin{aligned}\|\ell_{i,\nu} - s_{i,\nu}\|_{L_p(\omega_i)}^p &= \sum_{j=1}^{N_2} \int_{\omega_{ij}^{(\nu)}} |a_{i,\nu}(x_\nu - x_{i,\nu} - b_j)|^p \, dx \\ &= \frac{|a_{i,\nu}|^p}{m^{d-1}} \sum_{j=1}^{N_2} \int_0^{\frac{1}{mN_2}} u^p \, du = \Big(\frac{1}{mN_2}\Big)^p \frac{|a_{i,\nu}|^p}{m^d(p+1)}.\end{aligned}$$

Observe that

$$\frac{|a_{i,\nu}|^p}{m^d} = |\omega_i| \Big| \frac{1}{|\omega_i|} \int_{\omega_i} D_{x_\nu} f(x) \, dx \Big|^p \le \int_{\omega_i} |D_{x_\nu} f(x)|^p \, dx.$$

By setting

$$s = \sum_{i=1}^{N_1} \Big(c_i + \sum_{\nu=1}^{d} s_{i,\nu} \Big) \chi_{\omega_i},$$

we obtain a function $s \in S_1(\mathcal{P}_m)$ satisfying

$$\begin{aligned}\|\ell - s\|_p &= \Big(\sum_{i=1}^{N_1} \Big\| \sum_{\nu=1}^{d} (\ell_{i,\nu} - s_{i,\nu}) \Big\|^p_{L_p(\omega_i)}\Big)^{\frac{1}{p}} \\ &\le \Big(\sum_{i=1}^{N_1} \Big(\sum_{\nu=1}^{d} \|\ell_{i,\nu} - s_{i,\nu}\|_{L_p(\omega_i)}\Big)^p\Big)^{\frac{1}{p}} \\ &\le \sum_{\nu=1}^{d} \Big(\sum_{i=1}^{N_1} \|\ell_{i,\nu} - s_{i,\nu}\|^p_{L_p(\omega_i)}\Big)^{\frac{1}{p}},\end{aligned}$$

by the triangle inequalities for both integral and discrete p-norm. In view of the estimates given above,

$$\begin{aligned}\|\ell - s\|_p &\le \frac{(p+1)^{-1/p}}{mN_2} \sum_{\nu=1}^{d} \Big(\sum_{i=1}^{N_1} \frac{|a_{i,\nu}|^p}{m^d}\Big)^{\frac{1}{p}} \\ &\le \frac{1}{mN_2} \sum_{\nu=1}^{d} \Big(\sum_{i=1}^{N_1} \int_{\omega_i} |D_{x_\nu} f(x)|^p \, dx\Big)^{\frac{1}{p}} \\ &= \frac{1}{mN_2} |f|_{W^1_p(\Omega)}. \end{aligned} \tag{9}$$

Since $N_2 = m$ and $m^{-2} = \big(\frac{|\mathcal{P}_m|}{d}\big)^{\frac{-2}{d+1}}$, the bound (7) with constant $C = d^{\frac{2}{d+1}} \max\{2d\rho_d^2, 1\}$ is obtained by combining (8) and (9). □

3 Sums of piecewise linear polynomials

In this section we approximate the function by using a sum of piecewise linear polynomials over several overlaying partitions of Ω.

Algorithm 3. *Assume $f \in W^2_1(\Omega)$, $\Omega = (0,1)^d$. Split Ω into $N_1 = m^d$, $m \in \mathbb{Z}_+$, subcubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$, whose edges are parallel to the coordinate axes. For each $i = 1, \ldots, N_1$, let H_i be the average Hessian matrix of f over ω_i,*

$$H_i = \left[\frac{1}{2|\omega_i|} \int_{\omega_i} D_{x_\nu x_\mu} f(x) \, dx\right]_{\nu,\mu=1,\ldots,d}.$$

and let $\sigma_{i,\nu}$, $\nu = 1, \ldots, d$, be unit eigenvectors of H_i. For each $\nu = 1, \ldots, d$, define $\Delta^{(\nu)}$ by splitting each ω_i into N_2 slices $\omega^{(\nu)}_{ij}$, $j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the eigenvector $\sigma_{i,\nu}$. Set $\mathcal{P} = \{\Delta^{(1)}, \ldots, \Delta^{(d)}\}$. Then $|\Delta^{(\nu)}| = N_1 N_2$ for each $\nu = 1, \ldots, d$ and $|\mathcal{P}| = dN_1N_2$.

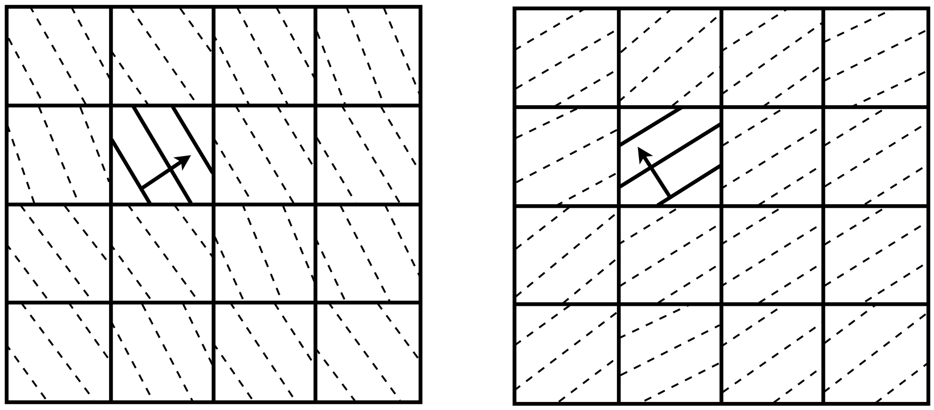

Figure 2: Partitions $\Delta^{(1)}$ (left) and $\Delta^{(2)}$ (right) obtained from Algorithm 3 ($d = 2$ and $N_2 = m = 4$).

Partitions $\Delta^{(1)}, \Delta^{(2)}$ of Algorithm 3 in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Figure 2. The splitting directions on each subcube ω_i are orthogonal to one of the eigenvectors of the average Hessian H_i.

Theorem 3. *Let $f \in W_p^3(\Omega)$, $\Omega = (0,1)^d$, for some $1 \le p \le \infty$. For any $m = 1, 2, \ldots$, generate the system of partitions $\mathcal{P}_m$ by using Algorithm 3 with $N_1 = m^d$ and $N_2 = \lceil m^{\frac{1}{2}} \rceil$. Then there exists a sum of piecewise linear functions $s_m \in S_2(\mathcal{P}_m)$ such that*

$$E_2(f, \mathcal{P}_m) \le \|f - s_m\|_p \le C_1 |\mathcal{P}_m|^{-6/(2d+1)} (|f|_{W_p^2(\Omega)} + |f|_{W_p^3(\Omega)}), \tag{10}$$

$$|f - s_m|_{W_p^1(\Omega)} \le C_2 |\mathcal{P}_m|^{-3/(2d+1)} (|f|_{W_p^2(\Omega)} + |f|_{W_p^3(\Omega)}), \tag{11}$$

where C_1, C_2 are constants depending only on d.

Proof. As in the proof of Theorem 2 we assume that $p < \infty$. The modifications needed in the case $p = \infty$ are obvious. Denote by Δ the partition of Ω into N_1 cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$. It follows from (3) that for each $i = 1, \ldots, N_1$ there exists a quadratic polynomial q_i such that

$$\|f - q_i\|_{L_p(\omega_i)} \le \rho_{d,3} \operatorname{diam}^3(\omega_i) |f|_{W_p^3(\omega_i)} \le \frac{d^{\frac{3}{2}} \rho_{d,3}}{m^3} |f|_{W_p^3(\omega_i)}, \tag{12}$$

$$|f - q_i|_{W_p^1(\omega_i)} \le \rho_{d,3} \operatorname{diam}^2(\omega_i) |f|_{W_p^3(\omega_i)} \le \frac{d \rho_{d,3}}{m^2} |f|_{W_p^3(\omega_i)}, \tag{13}$$

$$|f - q_i|_{W_p^2(\omega_i)} \le \rho_{d,3} \operatorname{diam}(\omega_i) |f|_{W_p^3(\omega_i)} \le \frac{d^{\frac{1}{2}} \rho_{d,3}}{m} |f|_{W_p^3(\omega_i)}. \tag{14}$$

We deduce from (12) that

$$\|f - \sum_{i=1}^{N_1} q_i \chi_{\omega_i}\|_p \le \frac{d^{\frac{3}{2}} \rho_{d,3}}{m^3} |f|_{W_p^3(\Omega)}. \tag{15}$$

Let $\tilde{q}_i(x) = x^T H_i x$ be the homogeneous quadratic polynomial whose Hessian matrix coincides with the average Hessian matrix H_i of f over ω_i, that is,

$$D_{x_\nu x_\mu} \tilde{q}_i = |\omega_i|^{-1} \int_{\omega_i} D_{x_\nu x_\mu} f(x)\, dx, \quad \nu, \mu = 1, \dots, d.$$

We can establish a relation between q_i and $\tilde{q}_i$ as follows. By using the Poincaré inequality (4), together with (14), we obtain

$$\begin{aligned} \|D_{x_\nu x_\mu}(\tilde{q}_i - q_i)\|_{L_p(\omega_i)} &\le \|D_{x_\nu x_\mu}(\tilde{q}_i - f)\|_{L_p(\omega_i)} + \|D_{x_\nu x_\mu}(f - q_i)\|_{L_p(\omega_i)} \\ &\le \rho_d \operatorname{diam}(\omega_i) \|\nabla(D_{x_\nu x_\mu} f)\|_{L_p(\omega_i)} + \rho_{d,3} \operatorname{diam}(\omega_i) |f|_{W_p^3(\omega_i)}. \end{aligned} \tag{16}$$

Let $i \in \{1, \dots, N_1\}$ be fixed, and let $(x_{0,1}, \dots, x_{0,d})$ denote the barycenter of ω_i. Consider the linear polynomial $\tilde{\ell}_i$ defined by

$$\tilde{\ell}_i = \tilde{c}_i + \sum_{\nu=1}^{d} \tilde{\ell}_{i,\nu}, \qquad \tilde{\ell}_{i,\nu} := \tilde{a}_{i,\nu}(x_\nu - x_{0,\nu}), \tag{17}$$

where

$$\tilde{a}_{i,\nu} = |\omega_i|^{-1} \int_{\omega_i} D_{x_\nu}(f - \tilde{q}_i)(x) dx, \quad \tilde{c}_i = |\omega_i|^{-1} \int_{\omega_i} (f - \tilde{q}_i)(x) dx. \tag{18}$$

Observe that $\int_{\omega_i} \tilde{\ell}_{i,\nu}(x) dx = 0$ for all $\nu = 1, \dots, d$. Hence, by using (4), we obtain

$$\begin{aligned} \|f - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} &= \|\big(f - \tilde{q}_i - \sum_{\nu=1}^{d} \tilde{\ell}_{i,\nu}\big) - \tilde{c}_i\|_{L_p(\omega_i)} \\ &\le \rho_d \operatorname{diam}(\omega_i) \|\nabla\big(f - \tilde{q}_i - \sum_{\nu=1}^{d} \tilde{\ell}_{i,\nu}\big)\|_{L_p(\omega_i)} \\ &\le \rho_d \operatorname{diam}(\omega_i) |f - \tilde{q}_i - \sum_{\nu=1}^{d} \tilde{\ell}_{i,\nu}|_{W_p^1(\omega_i)}. \end{aligned} \tag{19}$$

We shall estimate the seminorm in the above inequality. To this end, observe that for each $\nu = 1, \ldots, d$, the Poincaré inequality and (18) yield

$$\begin{aligned}\|D_{x_\nu}\big(f - \tilde{q}_i - \sum_{\mu=1}^{d} \tilde{\ell}_{i,\mu}\big)\|_{L_p(\omega_i)} &= \|D_{x_\nu}(f - \tilde{q}_i) - \tilde{a}_{i,\nu}\|_{L_p(\omega_i)} \\ &\leq \rho_d \operatorname{diam}(\omega_i) \|\nabla\big(D_{x_\nu}(f - \tilde{q}_i)\big)\|_{L_p(\omega_i)} \\ &\leq \rho_d \operatorname{diam}(\omega_i) |D_{x_\nu}(f - \tilde{q}_i)|_{W_p^1(\omega_i)}. \qquad (20)\end{aligned}$$

Now, for each $\mu = 1, \ldots, d$, by virtue of the definition of $\tilde{q}_i$, the Poincaré inequality implies that

$$\|D_{x_\mu x_\nu}(f - \tilde{q}_i)\|_{L_p(\omega_i)} \leq \rho_d \operatorname{diam}(\omega_i) \|\nabla(D_{x_\mu x_\nu} f)\|_{L_p(\omega_i)}. \qquad (21)$$

Combining (19), (20) and (21) we obtain

$$\|f - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} \leq \rho_d^3 \operatorname{diam}(\omega_i)^3 |f|_{W_p^3(\omega_i)}. \qquad (22)$$

Using the above estimation, together with (12), yields

$$\begin{aligned}\|q_i - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} &\leq \|q_i - f\|_{L_p(\omega_i)} + \|f - \tilde{q}_i - \tilde{\ell}_i\|_{L_p(\omega_i)} \\ &\leq \frac{d^{\frac{3}{2}}}{m^3}(\rho_{d,3} + \rho_d^3)|f|_{W_p^3(\omega_i)}. \qquad (23)\end{aligned}$$

Since $\tilde{c}_i$ is a constant, we have

$$|f - \tilde{q}_i - \tilde{\ell}_i|_{W_p^1(\omega_i)} = |f - \tilde{q}_i - \sum_{\nu=1}^{d} \tilde{\ell}_{i,\nu}|_{W_p^1(\omega_i)} \leq \rho_d^2 \operatorname{diam}(\omega_i)^2 |f|_{W_p^3(\omega_i)},$$

by virtue of (20) and (21). Combining this with (13) implies

$$\begin{aligned}|q_i - \tilde{q}_i - \tilde{\ell}_i|_{W_p^1(\omega_i)} &\leq |q_i - f|_{W_p^1(\omega_i)} + |f - \tilde{q}_i - \tilde{\ell}_i|_{W_p^1(\omega_i)} \\ &\leq \frac{d}{m^2}(\rho_{d,3} + \rho_d^2)|f|_{W_p^3(\omega_i)}. \qquad (24)\end{aligned}$$

For each $i = 1, \ldots, N_1$, the Hessian matrix H_i can be diagonalized into $H_i = U_i^T D_i U_i$ where U_i is an orthogonal matrix

$$U_i = [\sigma_{i,1} \cdots \sigma_{i,d}]^T,$$

and D_i is a diagonal matrix whose entries $\lambda_{i,1}, \ldots, \lambda_{i,d}$ are the eigenvalues of H_i. Then

$$\tilde{q}_i = \lambda_{i,1}\ell_1^2 + \cdots + \lambda_{i,d}\ell_d^2,$$

where

$$\ell_\nu(x) := \sigma_{i,\nu}^T x, \quad \nu = 1, \dots, d,$$

are linear polynomials. We have

$$|\lambda_{i,\nu}| \le \|H_i\|_\infty = \frac{1}{2|\omega_i|} \max_{1\le\gamma\le d} \sum_{\mu=1}^{d} \int_{\omega_i} |D_{x_\gamma x_\mu} f(x)|\, dx \le \frac{1}{2} |\omega_i|^{-1/p} |f|_{W_p^2(\omega_i)}.$$

Since $|\omega_i| = m^{-d}$, it follows that

$$|\lambda_{i,\nu}| \le \frac{m^{d/p}}{2} |f|_{W_p^2(\omega_i)}, \quad \nu = 1, \dots, d. \tag{25}$$

Given i, ν and j, the set $\omega_{ij}^{(\nu)}$ is contained between two hyperplanes $\ell_\nu(x) = c_j$ and $\ell_\nu(x) = c_j + \frac{w_{i,\nu}}{mN_2}$, where $w_{i,\nu}$ denotes the width of the unit cube in the direction $\sigma_{i,\nu}$. Clearly, $1 \le w_{i,\nu} \le \sqrt{d}$. We set

$$\bar{s}_{i,\nu} := \sum_{j=1}^{N_2} \lambda_{i,\nu} c_j (2\ell_\nu - c_j) \chi_{\omega_{ij}^{(\nu)}},$$

$$\bar{s}_i := \tilde{\ell}_i + \sum_{\nu=1}^{d} \bar{s}_{i,\nu}.$$

Then by using the orthogonal change of variables $y = \phi_{U_i}(x) := U_i x$ we obtain in view of (25),

$$\begin{aligned}
\|\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu}\|^p_{L_p(\omega_{ij}^{(\nu)})} &= \int_{\omega_{ij}^{(\nu)}} |\lambda_{i,\nu}(\ell_\nu(x) - c_j)^2|^p\, dx = \int_{\phi_{U_i}(\omega_{ij}^{(\nu)})} |\lambda_{i,\nu}(y_\nu - c_j)^2|^p\, dy \\
&\le \Big(\frac{\sqrt{d}}{m}\Big)^{d-1} \int_{c_j}^{c_j + \frac{w_{i,\nu}}{mN_2}} |\lambda_{i,\nu}(y_\nu - c_j)^2|^p\, dy_\nu \\
&\le \frac{d^{d/2}}{(2p+1)m^d N_2} \Big(\frac{d|\lambda_{i,\nu}|}{m^2 N_2^2}\Big)^p \\
&\le \frac{d^{d/2}}{(2p+1)N_2} \Big(\frac{d}{2m^2 N_2^2}\Big)^p |f|^p_{W_p^2(\omega_i)},
\end{aligned} \tag{26}$$

which implies

$$\|\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu}\|^p_{L_p(\omega_i)} = \sum_{j=1}^{N_2} \|\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu}\|^p_{L_p(\omega_{ij}^{(\nu)})} \le \frac{d^{d/2}}{2p+1} \Big(\frac{d}{2m^2 N_2^2}\Big)^p |f|^p_{W_p^2(\omega_i)}.$$

Hence

$$\|\tilde{q}_i+\tilde{\ell}_i-\bar{s}_i\|_{L_p(\omega_i)}=\Big\|\sum_{\nu=1}^{d}(\lambda_{i,\nu}\ell_\nu^2-\bar{s}_{i,\nu})\Big\|_{L_p(\omega_i)}\le\Big(\frac{d^{d/2}}{2p+1}\Big)^{\frac{1}{p}}\frac{d^2}{2m^2N_2^2}|f|_{W_p^2(\omega_i)},$$

from which it immediately follows that

$$\begin{aligned}\|\sum_{i=1}^{N_1}(\tilde{q}_i+\tilde{\ell}_i-\bar{s}_i)\chi_{\omega_i}\|_p&=\Big(\sum_{i=1}^{N_1}\|\tilde{q}_i+\tilde{\ell}_i-\bar{s}_i\|_{L_p(\omega_i)}^p\Big)^{\frac{1}{p}}\\&\le\Big(\frac{d^{d/2}}{2p+1}\Big)^{\frac{1}{p}}\frac{d^2}{2m^2N_2^2}|f|_{W_p^2(\Omega)}.\end{aligned}\tag{27}$$

Consider

$$s=\sum_{i=1}^{N_1}\bar{s}_i\chi_{\omega_i}.$$

Then $s\in S_2(\mathcal{P}_m)$. Since $m^{-3}\le(mN_2)^{-2}\le\big(\frac{|\mathcal{P}_m|}{d}\big)^{-6/(2d+1)}$, we now combine (15) with (23) and (27) to deduce that

$$\begin{aligned}\|f-s\|_p&\le\|f-\sum_{i=1}^{N_1}q_i\chi_{\omega_i}\|_p+\|\sum_{i=1}^{N_1}(q_i-\tilde{q}_i-\tilde{\ell}_i)\chi_{\omega_i}\|_p+\|\sum_{i=1}^{N_1}(\tilde{q}_i+\tilde{\ell}_i-\bar{s}_i)\chi_{\omega_i}\|_p\\&\le C_1|\mathcal{P}_m|^{-6/(2d+1)}\big(|f|_{W_p^3(\Omega)}+|f|_{W_p^2(\Omega)}\big),\end{aligned}$$

where $C_1=d^{\frac{6}{2d+1}}\Big(d^{\frac{3}{2}}(2\rho_{d,3}+\rho_d^3)+\Big(\frac{d^{d/2}}{2p+1}\Big)^{\frac{1}{p}}\frac{d^2}{2}\Big)$, and thereby proving (10).

For each $i=1,\dots,N_1$ we observe that

$$\begin{aligned}|f-\bar{s}_i|_{W_p^1(\omega_i)}\le 3^{1-\frac{1}{p}}\Big(\sum_{\nu=1}^{d}\int_{\omega_i}&\big(|D_{x_\mu}(f-q_i)(x)|^p\\&+|D_{x_\mu}(q_i-\tilde{q}_i-\tilde{\ell}_i)(x)|^p+|D_{x_\mu}(\tilde{q}_i+\tilde{\ell}_i-\bar{s}_i)(x)|^p\big)\,dx\Big)^{\frac{1}{p}}.\end{aligned}\tag{28}$$

For any $\mu,\nu=1,\dots,d$, denoting by $\sigma_{i,\nu}[\mu]$ the μ-th coordinate of the eigenvector $\sigma_{i,\nu}$, with $|\sigma_{i,\nu}[\mu]|\le 1$, we again use the orthogonal change of variables

$y = \phi_{U_i}(x)$ and (25) to show that

$$\begin{aligned}
\|D_{x_\mu}(\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu})\|^p_{L_p(\omega_{ij}^{(\nu)})} &= \int_{\omega_{ij}^{(\nu)}} 2|\lambda_{i,\nu}\sigma_{i,\nu}[\mu](\ell_\nu(x) - c_j)|^p \, dx \\
&\le 2^p \int_{\phi_{U_i}(\omega_{ij}^{(\nu)})} |\lambda_{i,\nu}(y_\nu - c_j)|^p \, dy \\
&\le 2^p \Big(\frac{\sqrt{d}}{m}\Big)^{d-1} \int_{c_j}^{c_j + \frac{w_{i,\nu}}{mN_2}} |\lambda_{i,\nu}(y_\nu - c_j)|^p \, dy_\nu \\
&\le \frac{2^p d^{d/2}}{(p+1)m^d N_2} \Big(\frac{\sqrt{d}|\lambda_{i,\nu}|}{mN_2}\Big)^p \\
&\le \frac{d^{d/2}}{(p+1)N_2} \Big(\frac{\sqrt{d}}{mN_2}\Big)^p |f|^p_{W_p^2(\omega_i)}, \qquad (29)
\end{aligned}$$

which implies

$$\begin{aligned}
\|D_{x_\mu}(\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu})\|^p_{L_p(\omega_i)} &= \sum_{j=1}^{N_2} \|D_{x_\mu}(\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu})\|^p_{L_p(\omega_{ij}^{(\nu)})} \\
&\le \frac{d^{d/2}}{p+1} \Big(\frac{\sqrt{d}}{mN_2}\Big)^p |f|^p_{W_p^2(\omega_i)}.
\end{aligned}$$

Hence, for each $\mu = 1, \ldots, d$, we have

$$\begin{aligned}
\|D_{x_\mu}(\tilde{q}_i + \tilde{\ell}_i - \bar{s}_i)\|^p_{L_p(\omega_i)} &= \Big\| \sum_{\nu=1}^{d} D_{x_\mu}(\lambda_{i,\nu}\ell_\nu^2 - \bar{s}_{i,\nu}) \Big\|^p_{L_p(\omega_i)} \\
&\le \frac{d^{d/2}}{p+1} \Big(\frac{d^{3/2}}{mN_2}\Big)^p |f|^p_{W_p^2(\omega_i)}. \qquad (30)
\end{aligned}$$

Combining (13), (24) and (30) shows that

$$\begin{aligned}
|f - \bar{s}_i|^p_{W_p^1(\omega_i)} \le & 3^{p-1}\Big(\big(\frac{d\rho_{d,3}}{m^2}\big)^p |f|^p_{W_p^3(\omega_i)} + d^p \big(\frac{\rho_{d,3} + \rho_d^2}{m^2}\big)^p |f|^p_{W_p^3(\omega_i)} \\
& + \frac{d^{d/2}}{p+1} \Big(\frac{d^{5/2}}{mN_2}\Big)^p |f|^p_{W_p^2(\omega_i)} \Big),
\end{aligned}$$

where, since $m^{-2} \le (mN_2)^{-1} \le \big(\frac{|\mathcal{P}_m|}{d}\big)^{-3/(2d+1)}$,

$$|f - s|_{W_p^1(\Omega)} \le C_2 |\mathcal{P}_m|^{-3/(2d+1)} \big(|f|_{W_p^3(\Omega)} + |f|_{W_p^2(\Omega)}\big),$$

with $C_2 = 3d^{\frac{3}{2d+1}} \Big(d\rho_{d,3} + d(\rho_{d,3} + \rho_d^2) + \Big(\frac{d^{d/2}}{p+1}\Big)^{\frac{1}{p}} d^{\frac{3}{2}} \Big)$, and (11) is proved. □

4 Sums of piecewise linear polynomials with fixed directions

In the previous section, the splitting directions in Algorithm 3 depend on the eigenvectors of the average Hessian matrices of f. In this section, we present another method where the splitting directions are independent of the function.

Lemma 2. *Any homogeneous quadratic polynomial q can be represented as a linear combination of $\binom{d+1}{2}$ quadratic ridge functions*

$$q = \sum_{\nu=1}^{d} a_\nu x_\nu^2 + \sum_{\nu=1}^{d-1} \sum_{\mu=\nu+1}^{d} b_{\nu\mu}(x_\nu + x_\mu)^2, \tag{31}$$

where

$$a_\nu = \frac{1}{2} D_{x_\nu x_\nu} q - \frac{1}{2} \sum_{\mu \neq \nu} D_{x_\nu x_\mu} q, \quad b_{\nu\mu} = \frac{1}{2} D_{x_\nu x_\mu} q. \tag{32}$$

Proof. To prove the first statement we just need to find this representation for all quadratic monomials. For $q = x_\nu^2$, we simply take $a_\nu = 1$ and set all other coefficients to zero. Moreover,

$$2x_\nu x_\mu = (x_\nu + x_\mu)^2 - x_\nu^2 - x_\mu^2,$$

so that for $q = x_\nu x_\mu$ with $\nu \neq \mu$ we can use $b_{\nu\mu} = \frac{1}{2}$, $a_\nu = a_\mu = -\frac{1}{2}$. The formulas (32) follow by a simple computation. □

In the algorithm below, in contrast to Algorithm 3, the splitting directions of the macro-cells ω_i are independent of f.

Algorithm 4. *Split $\Omega = (0,1)^d$ into $N_1 = m^d$, $m \in \mathbb{Z}_+$, cubes $\omega_1, \dots, \omega_{N_1}$ of edge length $1/m$, whose edges are parallel to coordinate axes. For each $\nu = 1, \dots, d$, define $\Delta^{(\nu)}$ by splitting each ω_i into N_2 slices $\omega_{ij}^{(\nu)}$ $j = 1, \dots, N_2$, by equidistant hyperplanes orthogonal to the x_ν-axis. For each pair $\{\nu, \mu\} \subset \{1, \dots, d\}$, $\nu \neq \mu$, define $\Delta^{(\nu,\mu)}$ by splitting each ω_i into N_2 slices $\omega_{ij}^{(\nu,\mu)}$, $j = 1, \dots, N_2$, by equidistant hyperplanes parallel to the subspace defined by $x_\nu + x_\mu = 0$. Set $\mathcal{P} = \{\Delta^{(1)}, \dots, \Delta^{(d)}, \Delta^{(1,2)}, \dots, \Delta^{(1,d)}, \dots, \Delta^{(d-1,d)}\}$. Then $|\Delta^{(\nu)}| = |\Delta^{(\nu,\mu)}| = N_1 N_2$ for all $\nu, \mu = 1, \dots, d$ and $|\mathcal{P}| = \binom{d+1}{2} N_1 N_2$.*

Partitions $\Delta^{(1)}, \Delta^{(2)}$ and $\Delta^{(1,2)}$ in the case $d = 2$ and $N_2 = m = 4$ are illustrated in Figure 3.

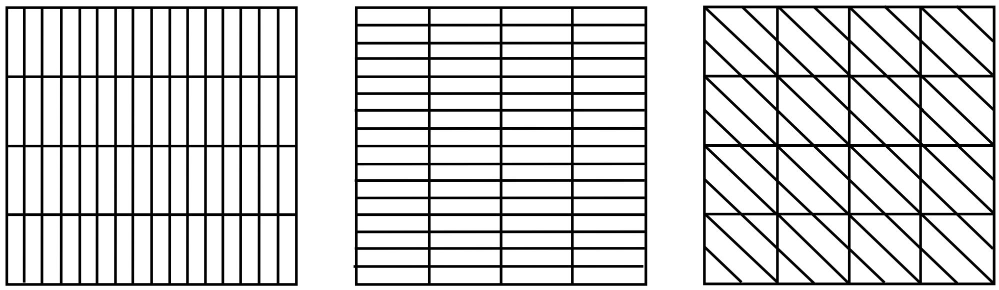

Figure 3: Partitions $\Delta^{(1)}, \Delta^{(2)}$ and $\Delta^{(1,2)}$ obtained from Algorithm 4 ($d = 2$ and $N_2 = m = 4$).

Theorem 4. *Let $f \in W_p^3(\Omega)$, $\Omega = (0,1)^d$, for some $1 \le p \le \infty$. For any $m = 1, 2, \ldots$, generate the system of partitions $\mathcal{P}_m$ by using Algorithm 4 with $N_1 = m^d$ and $N_2 = \lceil m^{\frac{1}{2}} \rceil$. Then there exists a sum of piecewise linear functions $s_m \in S_2(\mathcal{P}_m)$ such that*

$$E_2(f, \mathcal{P}_m) \le \|f - s_m\|_p \le C_1 |\mathcal{P}_m|^{-6/(2d+1)} (|f|_{W_p^2(\Omega)} + |f|_{W_p^3(\Omega)}), \tag{33}$$

$$|f - s_m|_{W_p^1(\Omega)} \le C_2 |\mathcal{P}_m|^{-3/(2d+1)} (|f|_{W_p^2(\Omega)} + |f|_{W_p^3(\Omega)}), \tag{34}$$

where C_1, C_2 are constants depending only on d.

Proof. As before we only consider the somewhat more difficult case when $p < \infty$ and leave the modifications needed for $p = \infty$ to the reader. Denote by Δ_m the partition of Ω into N_1 cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $1/m$. For each $i = 1, \ldots, N_1$, from (3) that there exists a quadratic polynomial q_i such that

$$\|f - q_i\|_{L_p(\omega_i)} \le \rho_{d,3} \operatorname{diam}(\omega_i)^3 |f|_{W_p^3(\omega_i)} \le \frac{d^{\frac{3}{2}} \rho_{d,3}}{m^3} |f|_{W_p^3(\omega_i)}, \tag{35}$$

$$|f - q_i|_{W_p^1(\omega_i)} \le \rho_{d,3} \operatorname{diam}(\omega_i)^2 |f|_{W_p^3(\omega_i)} \le \frac{d \rho_{d,3}}{m^2} |f|_{W_p^3(\omega_i)}, \tag{36}$$

$$|f - q_i|_{W_p^2(\omega_i)} \le \rho_{d,3} \operatorname{diam}(\omega_i) |f|_{W_p^3(\omega_i)} \le \frac{\sqrt{d} \rho_{d,3}}{m} |f|_{W_p^3(\omega_i)}. \tag{37}$$

By using (31) and the notation therein, let $q_i = q_i^{(1)} + q_i^{(2)}$ where

$$q_i^{(1)} = \sum_{\nu=1}^{d} a_\nu x_\nu^2, \quad \text{and} \quad q_i^{(2)} = \sum_{\nu=1}^{d} \sum_{\mu=\nu+1}^{d} b_{\nu\mu} (x_\nu + x_\mu)^2.$$

For fixed $\nu = 1, \ldots, d$ and $j = 1, \ldots, N_2$, there exists c_j such that the ν-th side of $\omega_{ij}^{(\nu)}$ is given by $[c_i, c_i + \frac{1}{mN_2}]$. Considering the linear polynomial

$s_i^{(1)} = \sum_{j=1}^{N_2}\sum_{\nu=1}^{d}(2a_\nu c_j x_\nu - a_\nu c_j^2)\chi_{\omega_{ij}^{(\nu)}}$, clearly

$$\begin{aligned}
\|q_i^{(1)} - s_i^{(1)}\|_{L_p(\omega_i)}^p &\le d^{p-1}\sum_{j=1}^{N_2}\sum_{\nu=1}^{d}\int_{\omega_{ij}^{(\nu)}}|a_\nu|^p|x_\nu - c_j|^{2p}dx \\
&= d^{p-1}\sum_{j=1}^{N_2}\sum_{\nu=1}^{d}\frac{1}{m^{d-1}}\int_{c_j}^{c_j+\frac{1}{mN_2}}|a_\nu|^p|x_\nu - c_j|^{2p}dx_\nu \\
&= d^{p-1}\sum_{j=1}^{N_2}\sum_{\nu=1}^{d}\frac{1}{m^{d-1}}\int_{0}^{\frac{1}{mN_2}}|a_\nu|^p|x_\nu|^{2p}dx_\nu \\
&= d^{p-1}\sum_{\nu=1}^{d}\frac{|a_\nu|^p}{(2p+1)m^d}\Big(\frac{1}{mN_2}\Big)^{2p}. \qquad (38)
\end{aligned}$$

By using the inequality between the arithmetic and the p-power means, together with (37), for each $i = 1, \ldots, N_1$, we have

$$\begin{aligned}
\sum_{\nu=1}^{d}\frac{|a_\nu|^p}{m^d} &= \sum_{\nu=1}^{d}\int_{\omega_i}\big|\tfrac{1}{2}D_{x_\nu x_\nu}q_i(x) - \tfrac{1}{2}\sum_{\mu\ne\nu}D_{x_\nu x_\mu}q_i(x)\big|^p\,dx \\
&\le 2^{p-2}\sum_{\nu=1}^{d}\int_{\omega_i}\Big(\big|D_{x_\nu x_\nu}(q_i - f)(x)\big|^p + \big|\sum_{\mu\ne\nu}D_{x_\nu x_\mu}(q_i - f)(x)\big|^p\Big)\,dx \\
&\quad + 2^p|f|_{W_p^2(\omega_i)}^p \\
&\le 2^p d^{p-1}\Big(\frac{\sqrt{d}\rho_{d,3}}{m}\Big)^p|f|_{W_p^3(\omega_i)}^p + 2^p|f|_{W_p^2(\omega_i)}^p, \qquad (39)
\end{aligned}$$

and (38) becomes

$$\|q_i^{(1)} - s_i^{(1)}\|_{L_p(\omega_i)}^p \le \Big(\frac{2d}{m^2N_2^2}\Big)^p\Big(d^{p-1}\Big(\frac{\sqrt{d}\rho_{d,3}}{m}\Big)^p|f|_{W_p^3(\omega_i)}^p + |f|_{W_p^2(\omega_i)}^p\Big). \qquad (40)$$

Given $\nu = 1, \ldots, d$ and $\mu = \nu+1, \ldots, d$, there exists b_j such that the ν-th side of $\omega_{ij}^{(\nu,\mu)}$ lies between the hyperplanes $x_\nu + x_\mu = b_j$ and $x_\nu + x_\mu = b_j + w$ where $0 < w \le \frac{\sqrt{d}}{mN_2}$. Consider the linear polynomial

$$s_i^{(2)} = \sum_{j=1}^{N_2}\sum_{\nu=1}^{d}\sum_{\mu=\nu+1}^{d}2b_j b_{\nu\mu}(x_\nu + x_\mu - b_j)\chi_{\omega_{ij}^{(\nu,\mu)}}.$$

By using the change of variable $X = x_\nu + x_\mu$ and $Y = x_\nu - x_\mu$, where

$b_j \le X \le b_j + w$ and the range of Y is at most $\frac{\sqrt{d}}{m}$, we have

$$\|q_i^{(2)} - s_i^{(2)}\|_{L_p(\omega_i)}^p \le d^{2p-2} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \sum_{\mu=\nu+1}^{d} \int_{\omega_{ij}^{(\nu,\mu)}} |b_{\nu\mu}|^p |x_\nu + x_\mu - b_j|^{2p}\, dx$$

$$\le d^{2p-2} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \sum_{\mu=\nu+1}^{d} \frac{|b_{\nu\mu}|^p}{m^{d-2}} \Big(\frac{\sqrt{d}}{m} \int_{b_j}^{b_j+w} |X - 2b_j|^{2p}\, dX \Big)$$

$$\le d^{2p-2} \sum_{j=1}^{N_2} \sum_{\nu=1}^{d} \sum_{\mu=\nu+1}^{d} \frac{\sqrt{d}|b_{\nu\mu}|^p}{m^{d-1}} \frac{1}{2p+1} \Big(\frac{\sqrt{d}}{mN_2} \Big)^{2p+1}. \quad (41)$$

By using the inequality between the arithmetic and the p-power means and (37), for each $i = 1, \dots, N_1$, we find that

$$\sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \frac{|b_{\nu\mu}|^p}{m^d} = \sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \int_{\omega_i} |\tfrac{1}{2} D_{x_\nu x_\mu} q_i(x)|^p dx$$

$$\le 2^{p-1} \sum_{\nu=1}^{d} \sum_{\mu=1}^{d} \int_{\omega_i} \big|\tfrac{1}{2} D_{x_\nu x_\mu}(q_i - f)\big|^p dx + \tfrac{1}{2}|f|_{W_p^2(\omega_i)}^p$$

$$\le \tfrac{1}{2} \Big(\frac{\sqrt{d}\rho_{d,3}}{m} \Big)^p |f|_{W_p^3(\omega_i)}^p + \tfrac{1}{2}|f|_{W_p^2(\omega_i)}^p. \quad (42)$$

Combining (42) and (41) yields

$$\|q_i^{(2)} - s_i^{(2)}\|_{L_p(\omega_i)}^p \le d^{3p-2} \Big(\frac{1}{m^2 N_2^2} \Big)^p \Big(\Big(\frac{\sqrt{d}\rho_{d,3}}{m} \Big)^p |f|_{W_p^3(\omega_i)}^p + |f|_{W_p^2(\omega_i)}^p \Big). \quad (43)$$

With $s_i = s_i^{(1)} + s_i^{(2)}$, combining (40) and (43) yields

$$\|q_i - s_i\|_{L_p(\omega_i)}^p \le 2^{p-1} \|q_i^{(1)} - s_i^{(1)}\|_{L_p(\omega_i)}^p + 2^{p-1} \|q_i^{(2)} - s_i^{(2)}\|_{L_p(\omega_i)}^p$$

$$\le \big(d^p + d^{3p-2}\big) \Big(\frac{2}{mN_2} \Big)^{2p} \Big(\Big(\frac{\sqrt{d}\rho_{d,3}}{m} \Big)^p + 1 \Big) \big(|f|_{W_p^3(\omega_i)}^p + |f|_{W_p^2(\omega_i)}^p \big). \quad (44)$$

The inequality $\max\{m^{-3}, (mN_2)^{-2}\} \le 4\binom{d+1}{2}^{6/(2d+1)} |\mathcal{P}_m|^{-6/(2d+1)}$ is easily provable. Considering $s = \sum_{i=1}^{N_1} s_i \chi_{\omega_i}$ and $q = \sum_{i=1}^{N_1} q_i \chi_{\omega_i}$, (35) and (44) imply

$$\|f - s\|_p \le \Big(\sum_{i=1}^{N_1} \|f - q_i\|_{L_p(\omega_i)}^p \Big)^{\frac{1}{p}} + \Big(\sum_{i=1}^{N_1} \|q_i - s_i\|_{L_p(\omega_i)}^p \Big)^{\frac{1}{p}}$$

$$\le \frac{d^{\frac{3}{2}} \rho_{d,3}}{m^3} |f|_{W_p^3(\Omega)} + \frac{4d^{3-2/p} + 6d}{m^2 N_2^2} \big(\sqrt{d}\rho_{d,3} + 1 \big) \big(|f|_{W_p^3(\Omega)} + |f|_{W_p^2(\Omega)} \big)$$

$$\le C_1 |\mathcal{P}_m|^{-6/(2d+1)} \big(|f|_{W_p^3(\Omega)} + |f|_{W_p^2(\Omega)} \big),$$

where $C_1 = 4\binom{d+1}{2}^{6/(2d+1)}\big(d^{\frac{3}{2}}\rho_{d,3} + 10d^3\big(\sqrt{d}\rho_{d,3}+1\big)\big)$, and the result (33) is proved.

For each $i = 1, \ldots, N_1$, by using the triangle inequality, we observe that

$$|q_i - s_i|^p_{W^1_p(\omega_i)} \le (2d)^{p-1}\sum_{\nu=1}^{d}\Big(\|D_{x_\nu}(q_i^{(1)} - s_i^{(1)})\|^p_{L_p(\omega_i)} + \|D_{x_\nu}(q_i^{(2)} - s_i^{(2)})\|^p_{L_p(\omega_i)}\Big).$$

On one hand, a direct computation shows that, for each $\nu = 1, \ldots, d$,

$$\|D_{x_\nu}(q_i^{(1)} - s_i^{(1)})\|^p_{L_p(\omega_i)} = \frac{2^p}{p+1}\frac{|a_\nu|^p}{m^d}\big(\frac{1}{mN_2}\big)^p. \tag{45}$$

On another hand, since for each $k = 1, \ldots, d$,

$$D_{x_k}\Big(\sum_{\nu=1}^{d}\sum_{\mu=\nu+1}^{d} b_{\nu\mu}(x_\nu + x_\mu)^2\Big) = \sum_{\mu\ne k}^{d} 2b_{k\mu}(x_k + x_\mu),$$

and

$$D_{x_k}\Big(\sum_{j=1}^{N_2}\sum_{\nu=1}^{d}\sum_{\mu=1}^{d} 2b_{\nu\mu}b_j(x_\nu + x_\mu) - b_{\nu\mu}b_j^2\Big) = \sum_{j=1}^{N_2}\sum_{\mu\ne k}^{d} 2b_{k\mu}b_j,$$

we deduce that

$$\begin{aligned}
\|D_{x_k}(q_i^{(2)} - s_i^{(2)})\|^p_{L_p(\omega_i)} &\le d^{p-1}\sum_{j=1}^{N_2}\sum_{\mu\ne k}^{d}|2b_{k\mu}|^p\int_{\omega_{ij}^{(k,\mu)}}|x_k + x_\mu - b_j|^p dx \\
&\le d^{p-1}\sum_{j=1}^{N_2}\sum_{\mu\ne k}^{d}|2b_{k\mu}|^p\Big(\frac{\sqrt{d}}{m^{d-1}}\int_{b_j}^{b_j+\frac{\sqrt{d}}{mN_2}}|X - b_j|^p\,dX\Big) \\
&= d^{p-1}\sum_{j=1}^{N_2}\sum_{\mu\ne k}^{d}|2b_{k\mu}|^p\frac{\sqrt{d}}{m^{d-1}}\frac{1}{p+1}\big(\frac{\sqrt{d}}{mN_2}\big)^{p+1} \\
&= \frac{d^{\frac{3p}{2}}2^p}{p+1}\sum_{\mu\ne k}^{d}\frac{|b_{k\mu}|^p}{m^d}\big(\frac{1}{mN_2}\big)^p,
\end{aligned} \tag{46}$$

by virtue of a change of variable $X = x_\nu + x_\mu$, $Y = x_\nu - x_\mu$ where $b_j \le X \le b_j + \frac{\sqrt{d}}{mN_2}$ and the range of Y not more that $\frac{\sqrt{d}}{m}$. From (45) and (46), together with (39) and (42), we find that

$$|q_i - s_i|^p_{W^1_p(\omega_i)} \le \big(\frac{1}{mN_2}\big)^p\Big(\frac{2^{p-1}d^{\frac{5p}{2}-1} + 2^{2p}}{p+1}\Big)\Big(\big(\frac{\sqrt{d}\rho_{d,3}}{m}\big)^p|f|^p_{W^3_p(\omega_i)} + |f|^p_{W^2_p(\omega_i)}\Big). \tag{47}$$

It is easy to show that $m^{-2} \le (mN_2)^{-1} \le 2\binom{d+1}{2}^{3/(2d+1)}|\mathcal{P}_m|^{-3/(2d+1)}$. We deduce from (36) and (47) that

$$\begin{aligned} |f-s|_{W_p^1(\Omega)} \le & \Big(2^{p-1}\sum_{i=1}^{N_1}\big(|f-q_i|^p_{W_p^1(\omega_i)} + |q_i - s_i|^p_{W_p^1(\omega_i)}\big)\Big)^{\frac{1}{p}} \\ \le & C_2|\mathcal{P}_m|^{-3/(2d+1)}\big(|f|_{W_p^3(\Omega)} + |f|_{W_p^2(\Omega)}\big), \end{aligned} \tag{48}$$

where $C_2 = 4\binom{d+1}{2}^{3/(2d+1)}\big(d^2\rho_{d,3} + (d^{\frac{5}{2}}+2)(\sqrt{d}\rho_{d,3}+1)\big)$, hence (34). □

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