Cubic Spline Interpolation on Nested Polygon Triangulations

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Abstract. We develop an algorithm for constructing Lagrange and Hermite interpolation sets for spaces of cubic C^1-splines on general classes of triangulations built up of nested polygons whose vertices are connected by line segments. Additional assumptions on the triangulation are significantly reduced compared to the special class given in [4]. Simultaneously, we have to determine the dimension of these spaces, which is not known in general. We also discuss the numerical aspects of the method.

§1. Introduction

In contrast to univariate splines, it is a non-trivial problem to construct even one single set of interpolation points for bivariate spline spaces. Such interpolation sets for $S_q^r(\triangle)$, the space of splines of degree q and smoothness r, were constructed for crosscut-partitions $\triangle$ (see the survey [9] and the references therein). For general triangulations $\triangle$, interpolation sets were constructed for $S_q^1(\triangle)$, $q \geq 4$ in [3].

The case $q = 3$ is much more complicated given that not even the dimension of $S_3^1(\triangle)$ is known for arbitrary triangulations $\triangle$. It is an open question whether the dimension of $S_3^1(\triangle)$ is equal to Schumaker's lower bound [12]. The aim of this paper is to investigate interpolation by $S_3^1(\triangle)$ for general classes of triangulations $\triangle$ consisting of nested polygons whose vertices are connected by line segments. Following a general principle of choosing locally interpolation points for $S_3^1(\triangle)$ by passing from triangle to triangle, we describe an inductive method for constructing point sets that admit unique Lagrange (respectively Hermite) interpolation by $S_3^1(\triangle)$ under certain assumptions on $\triangle$. Moreover, we prove that the dimension of these spaces is equal to Schumaker's lower bound.

In this way we obtain a class of triangulations $\triangle$ which is significantly larger than the special class described in [4]. Moreover, the methods of proof in this paper are different from those in [4]. It is important to note that

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triangulations of this type can be constructed starting from any given points in the plane, see [11].

The numerical examples (with up to 100,000 interpolation points) show that in order to obtain good approximations, it is desirable to subdivide some of the triangles. Our method of constructing interpolation points also works for these modified triangulations.

We note that our interpolation method can be used for the construction of smooth surfaces without involving any derivative data. For scattered data fitting, the needed Lagrange data are approximately computed by local methods. In contrast to the finite element methods for cubic splines, we do not need to subdivide all triangles by a Clough-Tocher split or use derivatives.

§2. Preliminaries

Let $\triangle$ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$. We denote by $S_3^1(\triangle) = \{s \in C^1(\Omega) : \; s|_T \in \Pi_3, \; T \in \triangle\}$ the space of bivariate splines of degree 3 and smoothness 1 (with respect to $\triangle$). Here $\Pi_3 = \mathrm{span}\,\{x^\nu y^\mu : \nu, \mu \geq 0, \, \nu + \mu \leq 3\}$ denotes the space of bivariate polynomials of total degree 3.

We investigate the following interpolation problem. Construct a set $\{z_1, \ldots, z_N\}$ in Ω, where $N = \dim S_3^1 = (\triangle)$, such that for each function $f \in C(\Omega)$, a unique spline $s \in S_3^1(\triangle)$ exists such that $s(z_i) = f(z_i)$, $i = 1, \ldots, N$. Such a set $\{z_1, \ldots, z_N\}$ is called a Lagrange interpolation set for $S_3^1(\triangle)$. If also partial derivatives of f are involved, then we speak of a Hermite interpolation set for $S_3^1(\triangle)$.

In contrast to [4], we will use *Bernstein-Bézier techniques* [2,5]. Given a spline $s \in S_3^1(\triangle)$, we consider the following representation of the polynomial pieces $p = s|_T \in \Pi_3$ on the triangle $T \in \triangle$ with vertices v_1, v_2, v_3,

$$p(x,y) = \sum_{\nu+\mu+\sigma=3} a_{\nu,\mu,\sigma}^{[T]} \tfrac{3!}{\nu!\mu!\sigma!} \Phi_1^\nu(x,y)\Phi_2^\mu(x,y)\Phi_3^\sigma(x,y), \; (x,y) \in T, \tag{1}$$

where $\Phi_l \in \Pi_1$, $l = 1,2,3$, is uniquely defined by $\Phi_l(v_k) = \delta_{k,l}$, $k = 1,2,3$. This representation of p is called Bernstein-Bézier representation of p, the real numbers $a_{\nu,\mu,\sigma}^{[T]}$ are called the Bernstein-Bézier coefficients of p, and $\Phi_l(x,y)$, $l = 1,2,3$, are the barycentric coordinates (w.r.t. T) of $(x,y) \in T$.

Definition 1. *A set $A \subset \{(\nu, \mu, \sigma, T) : \nu + \mu + \sigma = 3, \; T \in \triangle\}$ is called an* admissible set *for $S_3^1(\triangle)$ if for every choice of coefficients $a_{\nu,\mu,\sigma}^{[T]}$, $(\nu, \mu, \sigma, T) \in A$, a unique spline $s \in S_3^1(\triangle)$ exists with these coefficients in the above Bernstein-Bézier representation.*

The above Bernstein-Bézier form can be used to express smoothness conditions of polynomial pieces on adjacent triangles T_1, T_2 with vertices v_1, v_2, v_3, respectively v_1, v_2, v_4 (cf. [2,5]).

Theorem 2. *Let s be a piecewise cubic polynomial function defined on $T_1 \cup T_2$. Then $s \in S_3^1(\{T_1, T_2\})$ iff $a_{\nu,\mu,0}^{[T_2]} = a_{\nu,\mu,0}^{[T_1]}$, $\nu + \mu = 3$, and $a_{\nu,\mu,1}^{[T_2]} = a_{\nu+1,\mu,0}^{[T_1]}\Phi_1(v_4) + a_{\nu,\mu+1,0}^{[T_1]}\Phi_2(v_4) + a_{\nu,\mu,1}^{[T_1]}\Phi_3(v_4)$, $\nu + \mu = 2$.*

For later use, we also mention here the following relations between the Bernstein-Bézier coefficients of a cubic polynomial p in the representation (1) and its partial derivatives at v_1 in direction of a unit vector parallel to the edge $e = [v_1, v_2]$, denoted by $\frac{\partial}{\partial e}$.

$$
\begin{aligned}
&a^{[T]}_{3,0,0} = p(v_1), \quad a^{[T]}_{2,1,0} = p(v_1) + \frac{1}{3}\frac{\partial p(v_1)}{\partial e}\|v_1 - v_2\|_2, \\
&a^{[T]}_{1,2,0} = p(v_1) + \frac{2}{3}\frac{\partial p(v_1)}{\partial e}\|v_1 - v_2\|_2 + \frac{1}{6}\frac{\partial^2 p(v_1)}{\partial e^2}\|v_1 - v_2\|_2^2, \\
&\frac{\partial p(v_1)}{\partial e} = \frac{3(a^{[T]}_{2,1,0} - a^{[T]}_{3,0,0})}{\|v_1 - v_2\|_2}, \quad \frac{\partial^2 p(v_1)}{\partial e^2} = \frac{6(a^{[T]}_{1,2,0} - 2a^{[T]}_{2,1,0} + a^{[T]}_{3,0,0})}{\|v_1 - v_2\|_2^2}.
\end{aligned} \tag{2}
$$

§3. Main Results

In this section, we state our main results on $S^1_3(\triangle)$, where $\triangle$ consists of nested polygons whose vertices are connected by line segments. We first define this class of triangulations. Then, we determine the dimension and construct interpolation sets for the corresponding spline space. Moreover, we show that this dimension is equal to Schumaker's lower bound [12]. Finally, we discuss a property of $\triangle$, which is essential for the local construction of interpolation points.

First, we describe triangulations of nested polygons and decompose the domain into finitely many subsets needed in our construction of interpolation points.

Triangulations of Nested Polygons. We consider the following general type of triangulation $\triangle$. Let $P_0, P_1, \ldots, P_k$ be a sequence of closed simple polygonal lines, and let Ω_μ be the closed (not necessarily convex) bounded polygon with boundary P_μ. Suppose that the polygons Ω_μ are *nested*, *i.e.*, $\Omega_{\mu-1} \subset \Omega_\mu$, $\mu = 0, \ldots, k$. The vertices of $\triangle$ are the vertices of P_μ, $\mu = 0, \ldots, k$, and one vertex inside P_0. The edges of $\triangle$ are the edges of P_μ, $\mu = 0, \ldots, k$, and additional line segments connecting the vertices of P_μ with the vertices of $P_{\mu+1}$, $\mu = 0, \ldots, k-1$. The resulting triangulation $\triangle$ of $\Omega := \Omega_k$ does not have vertices in the interior of $\Omega_{\mu+1} \setminus \Omega_\mu$, $\mu = 0, \ldots, k-1$, and does not have edges connecting two vertices of P_μ other than the edges of P_μ, see Figure 1.

Decomposition of the Domain. We decompose the domain Ω into finitely many sets $V_0 \subset V_1 \subset \cdots \subset V_m = \Omega$, where each set V_i, is the union of closed triangles of $\triangle, i = 0, \ldots, m$. Let V_0 be an arbitrary closed triangle of $\triangle$ in Ω_0. We define the sets $V_1 \subset \cdots \subset V_m$ by induction. Assuming V_{i-1} is defined, we choose a vertex v_i of $\triangle$ such that there exists at least one triangle of $\triangle$ with vertex v_i and a common edge with V_{i-1}. Let $T_{i,1}, \ldots, T_{i,n_i}$, $n_i \geq 1$, be all such triangles. We set $V_i = V_{i-1} \cup \overline{T}_{i,1} \cup \cdots \cup \overline{T}_{i,n_i}$, and denote by $\triangle_i = \{T \in \triangle : T \subset V_i\}$ the subtriangulation which corresponds to the set V_i.

The vertices $v_i, i = 1, \ldots, m$, are chosen as follows. After choosing V_0 to be an arbitrary closed triangle of $\triangle$ in Ω_0, we pass through the vertices of P_0

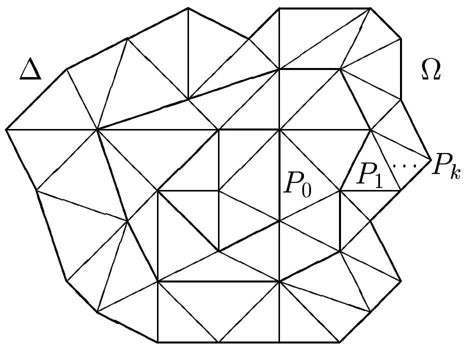

Fig. 1. Triangulation of nested polygons.

in clockwise order by applying the above rule. (It is clear that the choice of these vertices is unique after fixing the first vertex.) Now, we assume that we have passed through the vertices of $P_{\mu-1}$. We fix a vertex w_μ of P_μ that is connected with at least two vertices of $P_{\mu+1}$. Then w.r.t. clockwise order, we choose the first vertex of P_μ greater than w_μ which is connected with at least two vertices of $P_{\mu-1}$. Then we pass through the vertices of P_μ in clockwise order until w_μ^-, and pass through the vertices of P_μ in anticlockwise order until w_μ^+ by applying the above rule. (Here w_μ^+ denotes the vertex next to w_μ in clockwise order and w_μ^- denotes the vertex next to w_μ in anticlockwise order.) Finally, we choose the vertex w_μ. (It is clear that after fixing w_μ, the choice of the vertices on P_μ is unique.)

The construction of an admissible set for $S_3^1(\triangle)$ and the choice of interpolation points depend on the following properties of the triangulation $\triangle$.

Definition 3. *(1) An interior edge e with vertex v of the triangulation $\triangle$ is called* degenerate *at v if the edges with vertex v adjacent to e lie on a line. (2) An interior vertex v of $\triangle$ is called* singular *if v is a vertex of exactly four edges and these edges lie on two lines. (3) An interior vertex v of $\triangle$ on the boundary of a given subtriangulation $\triangle'$ of $\triangle$ is called* semi-singular of type 1 *w.r.t. $\triangle'$ if exactly one edge with endpoint v is not contained in $\triangle'$ and this edge is degenerate at v. (4) An interior vertex v of $\triangle$ on the boundary of a given subtriangulation $\triangle'$ of $\triangle$ is called* semi-singular of type 2 *w.r.t. $\triangle'$ if exactly two edges with endpoint v are not contained in $\triangle'$ and these edges are degenerate at v. (5) A vertex v of $\triangle$ is called* semi-singular *w.r.t. $\triangle'$ if v satisfies (3) or (4).*

In the following, we construct an admissible set and interpolation sets for $S_3^1(\triangle)$, where $\triangle$ is a nested-polygon triangulation.

Construction of an Admissible Set. First, we choose $\mathcal{A}_0 = \{(\nu,\mu,\sigma,V_0) : \nu+\mu+\sigma = 3\}$ and then, proceeding by induction, we successively add admissible points on $V_i \setminus V_{i-1}$, $i = 1,\ldots,m$. Assuming that an admissible set $\mathcal{A}_{i-1}$ on V_{i-1} has been constructed, we choose admissible points on $V_i \setminus V_{i-1}$ as follows. By the above decomposition of Ω, $V_i \setminus V_{i-1}$ is the union of consecutive triangles $T_{i,1},\ldots,T_{i,n_i}$ with vertex v_i and common edges with V_{i-1}. We denote the consecutive endpoints of these edges by $v_{i,0}, v_{i,1}, \ldots, v_{i,n_i}$, and the piecewise polynomials in the representation (1) on $T_{i,j}$ by $p_{i,j} \in \Pi_3$, $j = 1, \ldots n_i$, where the vertices of $T_{i,j}$ are ordered as follows: $v_i, v_{i,j}, v_{i,j+1}$. Furthermore, we denote by $e_{i,j}$ the edges $[v_{i,j}, v_i]$, $j = 0, \ldots, n_i$.

We need the following properties of the subtriangulation $\triangle_i = \{T \in \triangle : T \subset V_i\}$ at the vertices $v_{i,0}, \ldots, v_{i,n_i}$:

(a) $e_{i,j}$ is non-degenerate at $v_{i,j}$,

(b) $v_{i,j}$ is semi-singular w.r.t. $\triangle_i$. (This latter property is only relevant if $v_{i,j}$ lies on the boundary of $\triangle_i$, *i.e.*, for $j \in \{0, n_i\}$.)

For $j \in \{1, \ldots, n_i - 1\}$, we set $c_{i,j} = 1$ if (a) holds, and $c_{i,j} = 0$ otherwise. For $j \in \{0, n_i\}$, we set $c_{i,j} = 1$ if both (a) and (b) hold, and $c_{i,j} = 0$ otherwise. Moreover, we set $c_i = \sum_{j=0}^{n_i} c_{i,j}$, and assume $c_i \leq 3$, $i = 1, \ldots, n$.

Now, we construct the following admissible points on $V_i \setminus V_{i-1}$. If $c_i = 3$, then no point is chosen. If $c_i = 2$, then we choose $(3, 0, 0, T_{i,1})$. If $c_i = 1$, then we choose $(3, 0, 0, T_{i,1})$ and $(2, 0, 1, T_{i,j})$, where $e_{i,j}$ is an edge with $c_{i,j} = 0$. If $c_i = 0$, then we choose $(3, 0, 0, T_{i,1}), (2, 0, 1, T_{i,1})$ and $(2, 1, 0, T_{i,1})$. The admissible set $\mathcal{A}_i$ on V_i is obtained by adding these points to $\mathcal{A}_{i-1}$.

Construction of Interpolation Sets. We choose interpolation points in V_0 and then in $V_i \setminus V_{i-1}$, $i = 1, \ldots, m$, successively. In the first step, we choose in V_0 10 different points (respectively 10 Hermite interpolation conditions) which admit unique Lagrange interpolation (respectively Hermite interpolation) by the space Π_3. For example, for Lagrange interpolation, we may choose four parallel line segments l_ν in V_0 and ν different points on each $l_\nu, \nu = 1, 2, 3, 4$. Assuming that the interpolation points in V_{i-1} have already been chosen, we proceed to $V_i \setminus V_{i-1}$ as follows.

For Lagrange interpolation, we choose the following points in $V_i \setminus V_{i-1}$. If $c_i = 3$, then no point is chosen. If $c_i = 2$, then we choose v_i. If $c_i = 1$, then we choose v_i and one further point on some edge $e_{i,j}$ with $c_{i,j} = 0$. If $c_i = 0$, then we choose v_i and two further points on two different edges.

For Hermite interpolation, we require the following interpolation conditions for $s \in S_3^1(\triangle)$ at the vertex v_i. If $c_i = 3$, then no interpolation condition is required at v_i. If $c_i = 2$, then we require $s(v_i) = f(v_i)$. If $c_i = 1$, then we require $s(v_i) = f(v_i)$ and $\frac{\partial s}{\partial e_{i,j}}(v_i) = \frac{\partial f}{\partial e_{i,j}}(v_i)$, where $e_{i,j}$ is some edge with $c_{i,j} = 0$. If $c_i = 0$, then we require $s(v_i) = f(v_i)$, $\frac{\partial s}{\partial x}(v_i) = \frac{\partial f}{\partial x}(v_i)$ and $\frac{\partial s}{\partial y}(v_i) = \frac{\partial f}{\partial y}(v_i)$.

By the above construction, we obtain a set of points for Lagrange interpolation respectively a set of Hermite interpolation conditions.

Theorem 4. *Let $\triangle$ be a triangulation of nested polygons. If for all $i \in \{1, \ldots, m\}$, $c_i \leq 3$ and no vertex v_i is simultanously semi-singular (of type 2) w.r.t $\triangle_i$ and non-singular, then a unique spline in $S_3^1(\triangle)$ exists which satisfies the above Lagrange (respectively Hermite) interpolation conditions. In particular, the total number of interpolation conditions is equal to the dimension of $S_3^1(\triangle)$.*

Proof: First, we prove that the set constructed above is an admissible set for $S_3^1(\triangle)$. To this end, we show by induction that $\mathcal{A}_i$ is an admissible set for $S_3^1(\triangle)|_{\triangle_i} = \{s|_{\triangle_i} : s \in S_3^1(\triangle)\}$. This is clear for $i = 0$. Now, we assume that $\mathcal{A}_{i-1}$ is an admissible set for $S_3^1(\triangle)|_{\triangle_{i-1}}$, where $i \in \{1, \dots, m\}$, and consider V_i. For simplicity, we omit here the index i for $v_i, v_{i,j}, e_{i,j}, p_{i,j}, T_{i,j}$ and n_i. It follows from the induction hypothesis and Theorem 2 that the coefficients $a^{[T_j]}_{\nu,3-\nu-\sigma,\sigma}$, $\sigma = 0, \dots, 3-\nu$, $\nu = 0, 1$, of $p_j \in \Pi_3$, $j = 1, \dots, n$, on T_j, are uniquely determined. Moreover, if $c_{i,j} = 1$ for some $j \in \{1, \dots, n-1\}$, then it follows from Theorem 2 that the coefficient $a^{[T_j]}_{2,0,1}$ is uniquely determined.

In the following, we show that if $c_{i,0} = 1$, then the coefficient $a^{[T_1]}_{2,1,0}$ is uniquely determined. Let us consider the case where v_0 is semi-singular of type 2 w.r.t. $\triangle_i$. (The case that v_0 is semi-singular of type 1 w.r.t. $\triangle_i$ is analogous.) We denote by $\tilde{T}_l \in \triangle$, $l = 1, \dots, 3$, the triangles with vertex v_0 not contained in $\triangle_i$ in anticlockwise order, and by $\tilde{e}_l$ the common edge of $\tilde{T}_l$ and $\tilde{T}_{l+1}$, $l = 1, 2$. Since $\tilde{T}_3$ has a common edge with $\triangle_{i-1}$, it follows from Theorem 2 that the coefficient $\tilde{a}^{[\tilde{T}_3]}_{1,1,1}$ of $\tilde{p}_3 \in \Pi_3$ on $\tilde{T}_3$ is uniquely determined. Moreover, since $\tilde{e}_2$ and $\tilde{e}_1$ are degenerate at v_0, the coefficients $\tilde{a}^{[\tilde{T}_l]}_{1,1,1}$ of $\tilde{p}_l \in \Pi_3$ on $\tilde{T}_l$, $l = 1, 2$, are uniquely determined. Since e_0 is non-degenerate at v_0, it follows from Theorem 2 that the coefficient $a^{[T_1]}_{2,1,0}$ is uniquely determined. We note that since $\triangle$ is a nested-polygon triangulation, at least two triangles with vertex v_0 not contained in $\triangle_i$ exist. Therefore, if $c_{i,0} = 0$, then the coefficient $a^{[T_1]}_{2,1,0}$ is not yet determined.

Analogously as above, it can be shown that the coefficient $a^{[T_n]}_{2,0,1}$ is uniquely determined if $c_{i,n} = 1$. Otherwise, this coefficient is not yet determined.

Now, we consider the vertex v. The arguments below will show that we may assume that v is an interior point of $\triangle$. We denote by $T_{n+l} \in \triangle$, $l = 1, \dots, r$, $r \geq 3$, the triangles with vertex v not contained in $\triangle_i$ in anticlockwise order. Moreover, let the piecewise polynomials $p_{n+l} \in \Pi_3$, $l = 1, \dots, r$, on T_{n+l} in the representation (1) be given such that the first barycentric coordinate always corresponds to v. The above arguments show that exactly $c_i \leq 3$ coefficients of the set $\mathcal{C}_1 = \{a^{[T_l]}_{\nu,3-\nu-\sigma,\sigma} : \sigma = 0, \dots, 3-\nu, \nu = 2, 3, l = 1, \dots, n+r\}$ are uniquely determined. On the other hand, we construct $3 - c_i$ additional admissible points from $\mathcal{C}_1$ on $V_i \setminus V_{i-1}$. Now, it follows from the C^1-property at v and Theorem 2 that all coefficients from $\mathcal{C}_1$ are uniquely determined. By our method of passing through the vertices of $\triangle$, v is not semi-singular of type 1 w.r.t. $\triangle_i$. In particular, if $v = w_\mu$ for some $\mu \in \{0, \dots, k\}$. Moreover, by assumption v can be semi-singular of type 2 w.r.t. $\triangle_i$ only if v is singular. In this case, we have $r = 3$, and it follows from Theorem 3.3 in [13] that the coefficient $a^{[T_3]}_{1,1,1}$ is uniquely determined. Otherwise, if $r \geq 4$, then for some $l \in \{1, \dots, r-1\}$ one common edge of T_{n+l} and T_{n+l+1} is non-degenerate at v, and we can also proceed with our arguments.

Since all relevant differentiability conditions at the edges with endpoint v, respectively v_j, were involved, the above shows that $\mathcal{A}_i$ is an admissible set for $S_3^1(\triangle)|_{\triangle_i}$. Thus, the set $\mathcal{A}_m$ is an admissible set for $S_3^1(\triangle)$.

Therefore, the cardinality of $\mathcal{A}_m$ is equal to the dimension of $S_3^1(\triangle)$. By construction, it is evident that the number of Lagrange interpolation points, respectively the number of Hermite interpolation conditions coincides with this cardinality.

By an inductive argument, it follows from (2) that the Hermite interpolation conditions at v determine the Bernstein-Bézier coefficients of the admissible points chosen on $V_i \setminus V_{i-1}$. Analogously, the Lagrange interpolation conditions uniquely determine the interpolating spline on the edges of $V_i \setminus V_{i-1}$. Therefore, the interpolating spline is uniquely determined on all of $V_i \setminus V_{i-1}$. This completes the proof of Theorem 4. $\square$

For arbitrary triangulations, Schumaker [12] gave the following lower bound $L(\triangle)$ for the dimension of $S_3^1(\triangle)$,

$$L(\triangle) = 3V_B(\triangle) + 2V_I(\triangle) + \sigma(\triangle) + 1. \tag{3}$$

Here, $V_B(\triangle)$ is the number of boundary vertices of $\triangle$, $V_I(\triangle)$ is the number of interior vertices of $\triangle$ and $\sigma(\triangle)$ is the number of singular vertices of $\triangle$. For bounds on the dimension of bivariate spline spaces see also Manni [6].

Theorem 5. *If a triangulation $\triangle$ of nested polygons satisfies the hypotheses of Theorem 4, then the dimension of $S_3^1(\triangle)$ is equal to $L(\triangle)$.*

Proof: We have to show that the cardinality of $\mathcal{A}_m$ is equal to $L(\triangle)$. We prove this by induction. We set $\mathcal{S}(\triangle_0) = \emptyset$ and for $i \in \{1, \dots m\}$, we denote by $\mathcal{S}(\triangle_i)$ the set of boundary vertices w of $\triangle_i$ such that $w = v_{l,0}$ and $c_{l,0} = 1$ (respectively $w = v_{l,n_l}$ and $c_{l,n_l} = 1$) for some $l \in \{1, \dots, i\}$. Moreover, let $\tilde{\sigma}_i$ be the cardinality of $\mathcal{S}(\triangle_i)$ and a_i be the cardinality of $\mathcal{A}_i$. We will show that

$$L(\triangle_i) = a_i + \tilde{\sigma}_i, \qquad i = 0, \dots, m. \tag{4}$$

This is evident for $i = 0$. We assume that $L(\triangle_{i-1}) = a_{i-1} + \tilde{\sigma}_{i-1}$ for some $i \in \{1, \dots, m\}$ and consider V_i. We have $V_B(\triangle_i) = V_B(\triangle_{i-1}) - n_i + 2$, $V_I(\triangle_i) = V_I(\triangle_{i-1}) + n_i - 1$, $\sigma(\triangle_i) = \sigma(\triangle_{i-1}) + \gamma_i$, where γ_i is the number of singular vertices from the set $\{v_{i,j} : \ j = 1, \dots, n_i - 1\}$. Since $a_i = a_{i-1} + 3 - c_i$, it follows from the induction hypothesis and some elementary computations that

$$L(\triangle_i) = a_i + \tilde{\sigma}_{i-1} + c_i + \gamma_i - n_i + 1.$$

By our method of passing through the vertices of $\triangle$, it is evident that if $v_{i,0} = v_{i-1} \in \mathcal{S}(\triangle_i)$, then $v_{i,0} \notin \mathcal{S}(\triangle_{i-1})$. In the following, we show that if $v_{i,n_i} \in \mathcal{S}(\triangle_i)$, then $v_{i,n_i} \notin \mathcal{S}(\triangle_{i-1})$. First, let us assume that $v_{i,n_i} = v_{l,0}$ for some $l \in \{1, \dots, i-1\}$. If v_{i,n_i} is semi-singular of type 2 w.r.t. $\triangle_i$, then at least three edges of $\triangle$ not contained in $\triangle_l$ are attached to $v_{l,0}$. Hence, $c_{l,0} = 0$. If v_{i,n_i} is semi-singular of type 1 w.r.t. $\triangle_i$, then the edge e_{i,n_i} is non-degenerate at v_{i,n_i}, since $c_{i,n_i} = 1$. Therefore, $v_{l,0}$ is not semi-singular of type 2 w.r.t. $\triangle_l$. Again, $c_{l,0} = 0$ holds. The remaining case $v_{i,n_i} = v_{i-1,n_{i-1}}$, where $n_i = 1$, follows by the same arguments.

Now, we show for $j \in \{1, \ldots, n_i - 1\}$ that every non-singular vertex $v_{i,j}$ such that $e_{i,j}$ is degenerate at $v_{i,j}$ lies in $\mathcal{S}(\triangle_{i-1})$. First, we consider the case $j = 1$. Set $v_{i_1} = v_{i,1}$ and let $\tilde{e}_0$ be the edge that connects $v_{i,0}$ and $v_{i,1}$. We have to consider two cases.

Case 1. (The vertices $v_{i,1}$ and v_{i-2} are connected by an edge e.) If $\tilde{e}_0$ is non-degenerate at $v_{i,1}$ then $c_{i-1,n_{i-1}} = 1$. (In this case $v_{i,1}$ is semi-singular of type 1 w.r.t. $\triangle_{i-1}$.) Otherwise, since $v_{i,1}$ is non-singular, e is non-degenerate at $v_{i,1}$. Thus, $c_{i-2,n_{i-2}} = 1$. (In this case $v_{i,1}$ is semi-singular of type 2 w.r.t. $\triangle_{i-2}$.) We note that $v_{i,1}$ is not semi-singular w.r.t. $\triangle_{i_1+1}$, since at least three edges of $\triangle$ not contained in $\triangle_{i_1+1}$ are attached to $v_{i,1}$.

Case 2. (The vertices $v_{i,1}$ and v_{i-2} are not connected by an edge.) If $\tilde{e}_0$ is non-degenerate at $v_{i,1}$ then we also have $c_{i-1,n_{i-1}} = 1$. (In this case $v_{i,1}$ is semi-singular of type 1 w.r.t. $\triangle_{i-1}$.) We note that $v_{i,1}$ is not semi-singular w.r.t. $\triangle_{i_1+1}$, since $\tilde{e}_0$ is non-degenerate at $v_{i,1}$. Otherwise, let e be the edge that connects $v_{i,1}$ with v_{i_1+1}. Since $v_{i,1}$ is non-singular, e is non-degenerate at $v_{i,1}$. Thus, $c_{i_1+1,0} = 1$. (In this case $v_{i,1}$ is semi-singular of type 2 w.r.t. $\triangle_{i_1+1}$.) We note that in this case $v_{i,1}$ is semi-singular of type 1 w.r.t. $\triangle_{i-1}$, but $c_{i-1,n_{i-1}} = 0$.

Now, we consider the remaining case $j \in \{2, \ldots, n_i - 1\}$. Set $v_{i_j} = v_{i,j}$ and let e be the edge that connects $v_{i,j}$ with v_{i_j+1}. Since $v_{i,j}$ is non-singular, it follows that $v_{i,j}$ is not semi-singular of type 2 w.r.t. to $\triangle_{i_j}$. Therefore, e is non-degenerate at $v_{i,j}$. Hence, $c_{i_j+1,0} = 1$. (In this case $v_{i,j}$ is semi-singular of type 1 w.r.t. $\triangle_{i_j+1}$.) We note that in the case $j \in \{2, \ldots, n_i - 1\}$, by our method of passing through the vertices of $\triangle$, the value $c_{i_j-1,n_{i_j-1}}$ is not influenced by the geometrical properties of $\triangle$ at $v_{i,j}$.

The above proof now implies $\tilde{\sigma}_i = \tilde{\sigma}_{i-1} + c_i + \gamma_i - n_i + 1$, and therefore, (4) holds. Since $\tilde{\sigma}_m = 0$, we get $L(\triangle) = a_m$. This proves the theorem. □

In Theorem 4 we assume that for all $i \in \{1, \ldots, m\}$, no vertex v_i is simultaneously semi-singular (of type 2) w.r.t. $\triangle_i$ and non-singular. In the following, we show that this assumption is essential for the local construction of interpolation points.

Example 6. Let $v = v_i = (0,0)$, $v_5 = v_0 = v_{i,0} = (\gamma, 0)$, $\gamma < 0$, $v_1 = v_{i,1} = (\tau, m\tau)$, $\tau < 0$, $m > 0$, $v_2 = v_{i,2} = (0, \delta)$, $\delta < 0$, and set $v_3 = (\alpha, 0)$, $\alpha > 0, v_4 = (0, \beta)$, $\beta > 0$. Let v be connected with v_3 and v_4 and v_{l-1} be connected with v_l, $l = 1, \ldots, 5$. Then v is simultaneously semi-singular (of type 2) w.r.t. $\triangle_i$ and non-singular. Furthermore, we denote by T_l the triangle with vertices v, v_{l-1}, v_l and by $p_l \in \Pi_3$ the polynomial pieces on T_l, $l = 1, \ldots 5$, in the representation (1). We consider the set $\mathcal{C}_2 = \{a^{[T_l]}_{\nu,3-\nu-\sigma,\sigma},\ \sigma = 0, \ldots, 3-\nu, \nu = 1, \ldots, 3,\ l = 1, \ldots, 5\}$. For C^1-splines, it follows from Theorem 3.3 in [13] that each subset of $\mathcal{C}_2$ that uniquely determines all coefficients of $\mathcal{C}_2$ has cardinality 8 and contains the coefficients $a^{[T_l]}_{1,0,2}$, $l = 3, 4$. By the proof of Theorem 4, the coefficients $a^{[T_1]}_{1,2-\sigma,\sigma}$, $\sigma = 0, 1, 2$, and $a^{[T_2]}_{1,2-\sigma,\sigma}$, $\sigma = 1, 2$, are uniquely determined. If $e_{i,1}$ is non-degenerate at v_1, then in addition $a^{[T_1]}_{2,0,1}$ is uniquely determined. Otherwise, this coefficient is not determined. Hence, if $e_{i,1}$ is non-degenerate at v_1, then we have to choose exactly one additional coefficient to determine all coefficients of $\mathcal{C}_2$, and otherwise, we have to choose exactly two additional coefficients. We claim that in the latter case every choice of exactly two additional coefficients from the set $\{a^{[T_1]}_{3,0,0},\ a^{[T_1]}_{2,1,0},\ a^{[T_1]}_{2,0,1},\ a^{[T_2]}_{2,0,1}\}$ fails to determine all coefficients of $\mathcal{C}_2$.

Proof: Suppose that $e_{i,1}$ is degenerate at v_1 and choose, for example, $a^{[T_1]}_{2,1,0}$ and $a^{[T_1]}_{2,0,1}$. For simplicity, we set $a_1 = a^{[T_3]}_{2,1,0}$, $a_2 = a^{[T_3]}_{3,0,0}$, $a_3 = a^{[T_4]}_{2,0,1}$, $a_4 = a^{[T_4]}_{1,1,1}$, $a_5 = a^{[T_4]}_{2,1,0}$, $a_6 = a^{[T_3]}_{1,1,1}$, and assume that the remaining coefficients in $\mathcal{C}_2$ are zero. By Theorem 2,

$$a_3 = (\frac{\tau-\gamma}{m\tau\gamma}\beta + 1)a_2, \qquad a_4 = (1-\frac{\alpha}{\gamma})a_3, \qquad a_1 = (1-\frac{\delta}{\beta})a_2 + \frac{\delta}{\beta}a_3,$$
$$a_6 = (1-\frac{\delta}{\beta})a_5 + \frac{\delta}{\beta}a_4,\ a_5 = (1-\frac{\alpha}{\gamma})a_2,\ 0 = ((-\tau)(\frac{m}{\delta}+\frac{1}{\alpha})+1)a_1 + \frac{\tau}{\alpha}a_6.$$

Eliminating a_j, $j \in \{3,4,5\}$, yields $a_1 = (1+\delta\frac{\tau-\gamma}{m\tau\gamma})a_2$, $a_6 = (1-\frac{\alpha}{\gamma})(1+\delta\frac{\tau-\gamma}{m\tau\gamma})a_2$. By some elementary computations, we obtain for the determinant D of the corresponding system

$$D = \frac{(-1)(\tau(m\gamma+\delta)-\delta\gamma)^2}{m\tau\delta\gamma^2},$$

and it is easy to verify that $D = 0$ iff $e_{i,1}$ is degenerate at v_1. Other choices of exactly two additional coefficients from the set $\{a^{[T_1]}_{3,0,0},\ a^{[T_1]}_{2,1,0},\ a^{[T_1]}_{2,0,1},\ a^{[T_2]}_{2,0,1}\}$ can be examined in the same way, which proves our claim. $\square$

Note that if $e_{i,1}$ is non-degenerate at v_1, then every choice of exactly one additional coefficient in the set $\{a^{[T_1]}_{3,0,0},\ a^{[T_1]}_{2,1,0},\ a^{[T_2]}_{2,0,1}\}$ determines all coefficients in $\mathcal{C}_2$.

We finally discuss some numerical aspects of our scheme. A method for constructing nested polygon triangulations $\triangle$ of given points in the plane which satisfy the conditions of Theorem 4 was developed in [11]. Our numerical tests show that in order to obtain good approximations, it is necessary to subdivide some of the triangles (for details see [10,11]). Meanwhile, we have computed such examples with a high number of interpolation conditions. We only mention here that, for example, Lagrange respectively Hermite interpolation of Franke's test function by cubic C^1-splines with 118,822 interpolation conditions yields an error of $4.66902 * 10^{-6}$ in the uniform norm.

References

1. Alfeld, P., B. Piper, and L. L. Schumaker, An explicit basis for C^1 quartic bivariate splines, SIAM J. Numer. Anal. **24**, (1987), 891–911.

2. de Boor, C., B-form basics, in G. Farin (ed.), *Geometric Modeling*, SIAM, Philadelphia, 1987, 131–148.
3. Davydov, O., and G. Nürnberger, Interpolation by C^1 splines of degree $q \geq 4$ on triangulations, J. Comput. Appl. Math., to appear.
4. Davydov, O., G. Nürnberger, and F. Zeilfelder, Interpolation by cubic splines on triangulations, in C.K. Chui and L.L. Schumaker (eds.) *Approximation Theory IX*, Vanderbilt University Press, Nashville, 1998, 17–24.
5. Farin, G., Triangular Bernstein-Bézier patches, Comput. Aided Geom. Design **3** (1986) 83-127.
6. Manni, C., On the dimension of bivariate spline spaces over rectilinear partitions, Approx. Theory Appl. **7**, no. 2 (1991), 23-34.
7. Morgan, J., and R. Scott, A nodal basis for C^1 piecewise polynomials of degree $n \geq 5$, Math. Comp. **29** (1975), 736–740.
8. Nürnberger, G., *Approximation by Spline Functions*, SpringerVerlag, Berlin, Heidelberg, New York, 1989.
9. Nürnberger, G., O. V. Davydov, G. Walz, and F. Zeilfelder, Interpolation by bivariate splines on crosscut partitions, in G. Nürnberger, J. W. Schmidt and G. Walz (eds.), *Multivariate Approximation and Splines*, Birkhäuser Verlag, Basel, 1997, 189–204.
10. Nürnberger, G., and F. Zeilfelder, Lagrange interpolation by splines on triangulations, in R.H. Wang (ed.), *Proceedings of the Morningside Institute*, Peking, 1998.
11. Nürnberger, G., and F. Zeilfelder, Interpolation by spline spaces on classes of triangulations, J. Comput. Appl. Math., to appear.
12. Schumaker, L. L., Bounds on the dimension of spaces of multivariate piecewise polynomials, Rocky Mountain J. Math. **14**, (1984), 251–264.
13. Schumaker, L.L., Dual bases for spline spaces on a cell, Comput. Aided Geom. Design **5**, (1987), 277–284.

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