Interpolation and Almost Interpolation by Weak Chebyshev Spaces

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Abstract. Some new results on univariate interpolation by weak Chebyshev spaces, using conditions of Schoenberg-Whitney type and the concept of almost interpolation sets, are given.

§1. Introduction

Let U denote a finite-dimensional subspace of real-valued functions defined on some set K. We are interested in describing those configurations $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n = \dim U$, such that

$$\dim U_{|T} = s.$$

T is called an interpolation set (I-set) w.r.t. U. If $s = n$, then it is clearly equivalent to the condition that for any given data $\{y_1, \ldots, y_n\}$ there exists a unique $u \in U$ such that

$$u(t_i) = y_i, \qquad i = 1, \ldots, n.$$

It is well known that in the case of univariate polynomial spline spaces all interpolation sets can be characterized by the Schoenberg-Whitney condition (see, e.g., [3, 4]).

A new approach to multivariate interpolation has been found by Sommer and Strauss using the concept of almost interpolation. A set $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n$, is called an almost interpolation set (AI-set) w.r.t. U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, s$, there exist points $t'_i \in B_i$ such that $T' = \{t'_1, \ldots, t'_s\}$ is an I-set w.r.t. U. They have shown that for a wide class of generalized spline spaces defined on polyhedral partitions AI-sets can be characterized by conditions of Schoenberg-Whitney type (for detail see [3]).

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Davydov [1] has considered AI-sets in the case of an arbitrary topological space K. Using the concept of local dimension (Definition 1) he has shown that under some minor additional hypotheses on K, every subspace U has a spline-like structure and every AI-set w.r.t. U can be characterized by a Schoenberg-Whitney type condition.

In this paper we apply the concept of almost interpolation and local dimension to the case when K is a real subset and U denotes a weak Chebyshev space of dimension n; i.e., every $u \in U$ has at most $n-1$ sign changes.

§2. Characterizations of Interpolation Sets in the Weak Chebyshev Case

In the sequel we shall suppose that $K \subset \mathbb{R}$ and shall use the notations I-set and AI-set w.r.t. a space U, respectively, as defined above. We denote by $F(K)$ the linear space of all real-valued functions defined on K, and by $C(K)$, its subspace consisting of all continuous functions. Moreover, we denote the number of elements of a finite set M by card M.

We set, for a subspace U of $F(K)$,

$$Z(U) := \{t \in K : u(t) = 0 \text{ for all } u \in U\}.$$

For a space U and a subset M of K, we set

$$U(M) := \{u \in U : u = 0 \text{ on } M\}.$$

Moreover, we define for any subset $\{t_1, \dots, t_s\} \subset K$ such that $t_1 < \dots < t_s$,

$$\begin{gathered} t_{s+1} := t_1,\ [t_i, t_{i+1}] := \{t \in K : t_i \le t \le t_{i+1}\},\ i = 1, \dots, s-1, \\ [t_s, t_1] := \{t \in K : t \ge t_s \ \text{ or } \ t \le t_1\}. \end{gathered}$$

The following characterization of interpolation sets has been obtained in [2].

Theorem 1. *Let U be an n-dimensional weak Chebyshev subspace of $F(K)$. Moreover, suppose that $T = \{t_1, \dots, t_n\} \subset K \setminus Z(U)$ such that $t_1 < \dots < t_n$. The following conditions are equivalent:*

1) *T is an I-set w.r.t. U;*
2) *For all $P \subset \{1, \dots, n\}$,*

$$\operatorname{card}(T \cap \bigcup_{i \in P} [t_i, t_{i+1}]) \le \dim U_{|\bigcup_{i\in P}[t_i, t_{i+1}]}.$$

The assumption that U is a weak Chebyshev space cannot be removed.

Example 1. Let $K = [0, 3] \subset \mathbb{R}$ and assume that $U = \operatorname{span}\{u_1, u_2\}$, where $u_1 = 1$ on K and

$$u_2(t) = \begin{cases} 1 - t, & \text{if } 0 \le t \le 1 \\ 0, & \text{if } 1 < t < 2 \\ t - 2, & \text{if } 2 \le t \le 3. \end{cases}$$

Set $\tilde{u} = 1/2u_1 - u_2$. Then it is obvious that $\tilde{u}$ has two sign changes at $t_1 = 1/2$ and $t_2 = 5/2$, respectively. This implies that U fails to be a weak Chebyshev space and, in particular, that T fails to be an I-set w.r.t. U, where $T = \{t_1, t_2\}$. On the other hand, T satisfies condition 2 in Theorem 1.

We also need a spline-like behaviour of each finite-dimensional subspace U of $F(K)$ as shown in [1].

Definition 1. *Let t be any element of K. The* local dimension *of U on $\{t\}$ is defined by*

$$\phi(t) := \text{l-dim}_t\, U := \inf\{\dim U_{|B} : t \in B,\ B \,\text{open}\}.$$

It is easily seen that ϕ is an upper semicontinuous function on K. Moreover, some additional conditions hold.

Theorem 2. [3, p. 53] *Denote by $G_U \subset K$ the set of all points of continuity of local dimension. Then*

1) *G_U is an open and everywhere dense subset of K;*
2) *$G_U = \bigcup_{i\in I} K_i$, where K_i are disjoint connected components of G_U and I is a countable set;*
3) *If $Z(U) = \emptyset$ and U is a weak Chebyshev space, then $U_{|\text{int}\, K_i}$ is a Haar space for each $i \in I$ such that $\text{int}\, K_i \neq \emptyset$.*

Hence we consider the set $\{K_i\}_{i\in I}$ as a *partition* of K with *cells* K_i, and U as a "piecewise Haar" space.

Moreover, we need a "local" property of the elements of K. We say that a point $t \in K$ has V-property if either $(t-\varepsilon, t+\varepsilon) \cap K = \{t\}$ for some $\varepsilon > 0$ or

$$t = \sup\{x \in K : x < t\} = \inf\{x \in K : x > t\}.$$

We are now ready to state our first main result, which characterizes interpolation sets in terms of their locations with respect to the cells K_i.

Theorem 3. *Suppose that U denotes an n-dimensional weak Chebyshev subspace of $C(K)$. Let $K = \overline{\bigcup_{i\in I} K_i}$ be as above. Moreover, suppose that $T = \{t_1, \ldots, t_n\} \subset K \setminus Z(U)$ such that $t_1 < \cdots < t_n$, and every t_i has V-property. Then the following conditions are equivalent:*

1) *T is an I-set w.r.t. U;*
2) *For all $P \subset I$ we have*

$$\text{card}\,(T \cap M_P) \leq \dim U_{|M_P},$$

where $M_P := \overline{\bigcup_{i\in P} K_i}$.

Example 1 also shows that the assumption that U is a weak Chebyshev space cannot be removed from the statement of Theorem 3.

By definition it is obvious that every interpolation set represents an almost interpolation set. We are, therefore, interested in the converse; i.e., under what conditions an AI-set is already an I-set, and we obtain the following result for the case of weak Chebyshev spaces.

Theorem 4. *Let U denote an n-dimensional weak Chebyshev subspace of $F(K)$. Suppose that $T = \{t_1, \dots, t_s\} \subset G_U \setminus Z(U)$ such that $s \leq n$, and every t_i has V-property. The following conditions are equivalent:*

1) *T is an AI-set w.r.t. U;*
2) *T is an I-set w.r.t. U.*

This statement is no longer true if one omits the assumption that U is a weak Chebyshev space.

Example 2. Let us consider the subspace $U = \text{span}\,\{u_1, u_2\}$ of $C[0, 3]$ as has been defined in Example 1. Recall that U fails to be a weak Chebyshev space. Moreover, it is easily seen that the local dimension ϕ of U on $\{t\}$ is given by

$$\phi(t) = \begin{cases} 2 & \text{if } 0 \leq t \leq 1 \text{ and } 2 \leq t \leq 3 \\ 1 & \text{if } 1 < t < 2. \end{cases}$$

Set $T = \{t_1, t_2\}$ where $t_1 = 1/2$ and $t_2 = 5/2$. Then $T \subset G_U \setminus Z(U)$, and it is obvious that t_i has V-property for $i = 1, 2$. By the arguments in Example 1, we know that T fails to be an I-set w.r.t. U. On the other hand, T is an AI-set w.r.t. U, because it is easily verified that for each sufficiently small $\varepsilon > 0$ the set $T_\varepsilon = \{\tilde{t}_1, \tilde{t}_2\}$ is an I-set w.r.t. U, where $\tilde{t}_1 = t_1$ and $\tilde{t}_2 = t_2 + \varepsilon$, respectively.

Remark. Theorems 3 and 4 together extend a statement on interpolation by generalized splines [5, Theorem 4.6]. In fact, it is shown that every weak Chebyshev space satisfies a "weak SSW-property" in terminology of [5].

§3. Proofs

In this section we shall give the proofs of the Theorems 3 and 4. To prove the first one, we need the following lemma.

Lemma 1. *Let U denote an n-dimensional subspace of $C(K)$. Moreover, suppose that $T = \{t_1, \dots, t_s\} \subset K$, where $s \leq n$. Let $K = \overline{\bigcup_{i \in I} K_i}$ be decomposed as in Section 2. If for all $P \subset I$,*

$$\text{card}\,(T \cap M_P) \leq \dim U_{|M_P}, \tag{1}$$

where $M_P = \overline{\bigcup_{i \in P} K_i}$, and every point t_j has V-property, then for all $Q \subset \{1, \dots, s\}$,

$$\text{card}\,(T \cap R_Q) \leq \dim U_{|R_Q}, \tag{2}$$

where $R_Q = \bigcup_{j \in Q} [t_j, t_{j+1}]$.

Proof: Suppose that (2) fails. Then, for some $\tilde{Q} \subset \{1, \dots, s\}$, it follows that $\dim U_{|R_{\tilde{Q}}} < \text{card}\,(T \cap R_{\tilde{Q}})$. Let us consider the set

$$\tilde{P} = \{i \in I : K_i \cap R^o_{\tilde{Q}} \neq \emptyset\} \cup \{i \in I : K_i = \{t_j\} \,\text{or}\, K_i = \{t_{j+1}\} \,\text{for some}\, j \in \tilde{Q}\},$$

where

$$R^o_{\tilde{Q}} := \bigcup_{j\in\tilde{Q}} (t_j, t_{j+1}).$$

Then, because $\bigcup_{i\in I} K_i$ is everywhere dense in K, and every t_j has V-property, we have

$$R_{\tilde{Q}} \subset M_{\tilde{P}}. \tag{3}$$

In order to reach a contradiction to (1), it is now enough to show that

$$\dim U_{|M_{\tilde{P}}} = \dim U_{|R_{\tilde{Q}}}. \tag{4}$$

On the contrary, assume that (4) is not true. Then there exists $u \in U$ such that

$$u_{|R_{\tilde{Q}}} = 0 \quad \text{and} \quad u_{|M_{\tilde{P}}} \neq 0. \tag{5}$$

Since $u \in C(K)$, it follows from (5) that $u_{|\cup_{i\in\tilde{P}} K_i} \neq 0$ which implies that $u_{|K_{i_0}} \neq 0$ for some $i_0 \in \tilde{P}$. Let $x \in K_{i_0}$ be such that $u(x) \neq 0$. In view of (5), we have $x \notin R_{\tilde{Q}}$. Consequently, by the definition of $\tilde{P}$, we see that $K_{i_0} \cap R^o_{\tilde{Q}} \neq \emptyset$. Therefore, K_{i_0} also contains a point y such that $y \in R^o_{\tilde{Q}}$. Assume, without loss of generality, that $x < y$. Since K_{i_0} is connected, we have $(x, y) \subset K_{i_0}$. Moreover, there exists j_0 such that $x < t_{j_0} < y$ and $(t_{j_0}, y) \subset R^o_{\tilde{Q}}$. Then $u_{|(t_{j_0},y)} = 0$. However, by [3, Theorem 4.7], $U_{|(x,y)}$ is an almost Haar space. Hence, $u_{|(x,y)} = 0$, and, by continuity, $u(x) = 0$, a contradiction. $\square$

Proof of Theorem 3: Suppose first that T is an I-set w.r.t. U and

$$c := \text{card}\,(T \cap M_{\tilde{P}}) > \dim U_{|M_{\tilde{P}}} =: \tilde{c}$$

for some $\tilde{P} \subset I$. Thus we could interpolate arbitrary data $\{y_1, \ldots, y_c\}$ by $U_{|M_{\tilde{P}}}$ which contradicts $c > \tilde{c}$.

Suppose now that for all $P \subset I$, we have

$$\text{card}\,(T \cap M_P) \leq \dim U_{|M_P}.$$

Then, in view of Lemma 1, it is obvious that for all $Q \subset \{1, \ldots, n\}$,

$$\text{card}\,(T \cap R_Q) \leq \dim U_{|R_Q}.$$

Hence it follows from Theorem 1 that T is an I-set w.r.t. U. $\square$

To prove Theorem 4, we first introduce some notations. Suppose that U denotes a subspace of $F(A)$. A finite set $T = \{t_1, \ldots, t_s\}$ is said to be an NI-**set w.r.t.** U, if T fails to be an I-set w.r.t. U. Every minimal NI-set is called a C-**set w.r.t.** U.

It is easily verified that for every C-set $T = \{t_1, \ldots, t_s\}$ there exists a (up to a factor ± 1) unique **signature** $\varepsilon^T = \{\varepsilon^T_1, \ldots, \varepsilon^T_s\}$, $|\varepsilon^T_i| = 1$ for all i such that no function $u \in U$ satisfies $\varepsilon^T_i u(t_i) > 0$, $i = 1, \ldots, s$. By definition, we have $\dim U_{|T} = s - 1$.

The proof of Theorem 4 is based on the following three lemmas.

Lemma 2. *Let U denote an n-dimensional weak Chebyshev subspace of $F(K)$. Suppose that $T = \{t_1, \ldots, t_s\} \subset K$ is an NI-set w.r.t. U, where $t_1 < \ldots < t_s$, $s \le n$, and $X = \{x_1, \ldots, x_p\} \subset K$ is an I-set w.r.t. $U(T)$, where $x_1 < \ldots < x_p$ and $p = n - s + 1$. Then there does not exist any function $u \in U$ such that*

$$\varepsilon_i(X)u(t_i) > 0, \quad i = 1, \ldots, s, \tag{6}$$

where $\varepsilon(X) = \{\varepsilon_1(X), \ldots, \varepsilon_s(X)\}$ is defined by

$$\varepsilon_1(X) := 1, \quad \varepsilon_{i+1}(X) := (-1)^{\gamma_i+1}\varepsilon_i(X), \quad i = 1, \ldots, s-1 \tag{7}$$

with $\gamma_i :=$ card $(X \cap [t_i, t_{i+1}])$, $i = 1, \ldots, s-1$.

Proof: On the contrary, assume that there exists $u \in U$ such that (6) holds. Since X is an I-set w.r.t. $U(T)$, it is clear that $X \cap T = \emptyset$. Hence, for each $j \in \{1, \ldots, p\}$ there exists $i \in \{0, \ldots, s\}$ such that $x_j \in (t_i, t_{i+1})$. (We set $t_0 := -\infty$, $t_{s+1} := \infty$.) Let

$$\delta_j := \begin{cases} (-1)^{\nu_j}, & \nu_j = \text{card } (X \cap [x_j, t_1]) \quad \text{if } i = 0 \\ (-1)^{\nu_j}\varepsilon_i(X), & \nu_j = \text{card } (X \cap [t_i, x_j]) \quad \text{if } i \ge 1. \end{cases}$$

Because X is an I-set w.r.t. $U(T)$, there exists a function $v \in U(T)$ such that $v(x_j) = \delta_j$, $j = 1, \ldots, p$. Then for a sufficiently small $\alpha > 0$, the function $\tilde{u} := \alpha u + v$ has n sign changes on $T \cup X \subset K$ which contradicts the hypothesis on U to be a weak Chebyshev space. $\Box$

Lemma 3. *Let U denote an n-dimensional weak Chebyshev subspace of $F(K)$. Suppose that $T = \{t_1, \ldots, t_s\} \subset K$ is a C-set w.r.t. U, where $t_1 < \cdots < t_s$ with $s \le n$. Then*

$$U(T) = U(K \cap [t_1, t_s]) \oplus \bigoplus_{i=1}^{s-1} U(K \setminus (t_i, t_{i+1})). \tag{8}$$

Moreover, the signature ε^T is determined by the following equations

$$\varepsilon_i^T \varepsilon_{i+1}^T = (-1)^{\mu_i+1}, \quad i = 1, \ldots, s-1, \tag{9}$$

where $\mu_i := \dim U(K \setminus (t_i, t_{i+1}))$.

Proof: Because T is a C-set, we have $\dim U(T) = n - \dim U_{|T} = n - s + 1$. Hence there exists an I-set $X = \{x_1, \ldots, x_p\}$ w.r.t. $U(T)$ such that $p = n - s + 1$. In view of Lemma 2, the sign vector $\varepsilon(X) = \{\varepsilon_1(X), \ldots, \varepsilon_s(X)\}$ defined by (7) is a signature of T.

Let the functions $u_1, \ldots, u_p \in U(T)$ be given by the conditions $u_j(x_i) = \delta_{ij}$, $i, j = 1, \ldots, p$. If (8) is not true, then there exists $j_0 \in \{1, \ldots, p\}$ such that

$$u_{j_0} \notin U(K \cap [t_1, t_s]) \cup \bigcup_{i=1}^{s-1} U(K \setminus (t_i, t_{i+1})). \tag{10}$$

Let $i_0 \in \{0, \ldots, s\}$ such that $x_{j_0} \in (t_{i_0}, t_{i_0+1})$. By (10) we can find x'_{j_0} such that $u_{j_0}(x'_{j_0}) \neq 0$ and $x'_{j_0} \in (t_i, t_{i+1})$ with $i \neq i_0$ (and $i \notin \{0, s\}$ if $i_0 \in \{0, s\}$). Consider $X' := (X \setminus \{x_{j_0}\}) \cup \{x'_{j_0}\}$ which is evidently an I-set w.r.t. $U(T)$. Let $\varepsilon(X') = \{\varepsilon_1(X'), \ldots, \varepsilon_s(X')\}$ be defined by (7), replacing X by X'. It follows from Lemma 2 that $\varepsilon(X')$ is also a signature of T. However, it is clear that $\varepsilon(X') \notin \{\varepsilon(X), -\varepsilon(X)\}$. This contradicts the uniqueness of the signature up to a factor ± 1. Thus (8) holds.

In order to check (9), it is sufficient to notice that $\varepsilon^T = \pm\varepsilon(X)$ and, in view of (8), $\mu_i = \text{card}\,(X \cap [t_i, t_{i+1}])$, $i = 1, \ldots, s-1$. $\square$

Lemma 4. *Let K be a topological space, and let U denote a locally finite-dimensional subspace of $F(K)$ (i.e., for every $t \in K$ there exists a neighborhood $B(t)$ such that $U_{|B(t)}$ is finite-dimensional). Suppose that V is any linear subspace of U. Then*

$$G_V \supset G_U,$$

where G_U and G_V denote the sets of points of continuity of local dimension of U and V, respectively.

Proof: Define $\phi(t) := \text{l-dim}_t\, U$ and assume that $\bar{t} \in G_U$. Since ϕ takes only integer values, there exists an open set B, with $\bar{t} \in B \subset K$, such that $\phi(t) = c := \dim U_{|B}$ for all $t \in B$. Let $U_{|B} = V_{|B} \oplus \tilde{V}$. We set $\bar{\phi}(t) := \text{l-dim}_t\, V$ and $\tilde{\phi}(t) := \text{l-dim}_t\, \tilde{V}$, $t \in B$. Then $c = \dim V_{|B} + \dim \tilde{V}$, which implies that

$$\phi(t) \geq \bar{\phi}(t) + \tilde{\phi}(t), \quad t \in B.$$

We want to show the equality. Assume that $\phi(\tilde{t}) > \bar{\phi}(\tilde{t}) + \tilde{\phi}(\tilde{t})$ for some $\tilde{t} \in B$. Let $\hat{B} \subset B$, $\hat{B}$ open, $\tilde{t} \in \hat{B}$, such that $\bar{\phi}(\tilde{t}) = \dim V_{|\hat{B}}$ and $\tilde{\phi}(\tilde{t}) = \dim \tilde{V}_{|\hat{B}}$. Then $c = \dim U_{|\hat{B}} > \dim V_{|\hat{B}} + \dim \tilde{V}_{|\hat{B}}$, a contradiction.

Since both $\bar{\phi}$ and $\tilde{\phi}$ are upper semicontinuous, we have for some $\tilde{B} \subset B$, with $\bar{t} \in \tilde{B}$,

$$\bar{\phi}(t) \leq \bar{\phi}(\bar{t}), \quad \tilde{\phi}(t) \leq \tilde{\phi}(\bar{t}), \qquad t \in \tilde{B},$$

which, together with $\bar{\phi}(t) + \tilde{\phi}(t) = c$, $t \in B$, implies that each of $\bar{\phi}$ and $\tilde{\phi}$ is constant on $\tilde{B}$. In particular, $\bar{\phi}$ is continuous at $\bar{t}$, and hence $\bar{t} \in G_V$. $\square$

Proof of Theorem 4: Assume that T is an AI-set but fails to be an I-set w.r.t. U. Without loss of generality, let T be a C-set with $s \geq 2$, since $T \cap Z(U) = \emptyset$. Thus $\dim U_{|T} = s - 1$. It is easy to check that $T \subset Z(U(T))$ and $\dim U_{|T} = \dim U_{|Z(U(T))}$.

If $T \subset \operatorname{int} Z(U(T))$, then considering the local dimension of U on T, we obtain

$$\text{l-dim}_T U := \inf\{\dim U_{|B} : B \supset T,\ B \text{ open}\} \leq \dim U_{|\operatorname{int} Z(U(T))} = \dim U_{|T}$$

which implies that $\text{l-dim}_T U \leq s - 1 < \operatorname{card} T$ contradicting the hypothesis on T to be an AI-set. (We have used here a characterization of AI-sets given in [1, Theorem 3.3].)

Otherwise, there exists $i_0 \in \{1, \ldots, s\}$ such that $t_{i_0} \in \operatorname{bd} Z(U(T))$. This implies that t_{i_0} fails to be an isolated point of K, because otherwise $\{t_{i_0}\}$ would be open in K. Hence by the V-property, $t_{i_0} = \sup\{x \in K : x < t_{i_0}\} = \inf\{x \in K : x > t_{i_0}\}$. Then it follows from Lemma 3 that t_{i_0} is a point of discontinuity of the local dimension of $U(T)$. By Lemma 4, t_{i_0} is also a point of discontinuity of local dimension of U contradicting the hypothesis that $T \subset G_U$. $\square$

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