Interpolation by Cubic Splines on Triangulations

Oleg Davydov, Günther Nürnberger, and Frank Zeilfelder

Abstract. We describe an algorithm for constructing point sets which admit unique Lagrange and Hermite interpolation from the space $S_3^1(\Delta)$ of C^1 splines of degree 3 defined on a general class of triangulations Δ. The triangulations Δ consist of nested polygons whose vertices are connected by line segments. In particular, we have to determine the dimension of $S_3^1(\Delta)$ which is not known for arbitrary triangulations Δ. Numerical examples are given.

§1. Introduction

In the literature, point sets which admit unique Lagrange and Hermite interpolation from spaces $S_q^r(\Delta)$ of splines of degree q and smoothness r were constructed for crosscut partitions Δ, in particular for Δ^1- and Δ^2-partitions. Results on the approximation order of these interpolation methods were also proved. (Because of space limitations, we refer to the references of our paper [5] in this volume.) Hermite interpolation schemes for $S_q^1(\Delta), q \geq 5$, where Δ is an arbitrary triangulation, were given in [1, 3].

An inductive method for constructing Lagrange and Hermite interpolation points for $S_q^1(\Delta), q \geq 5$, where Δ is an arbitrary triangulation, was developed in [2]. Here, in each step, one vertex is added to the subtriangulation considered before. For $q = 4$, this method works under certain assumptions on Δ.

The most complex case is $q = 3$, since even the dimension of $S_3^1(\Delta)$ is not known for arbitrary triangulations Δ. In this paper, we develop Lagrange and Hermite interpolation methods for $S_3^1(\Delta)$. The triangulations Δ consist of nested polygons whose vertices are connected in a natural way. The interpolation points are constructed inductively by passing through the vertices of the nested polygons, where in contrast to [2], the choice of these vertices is unique.

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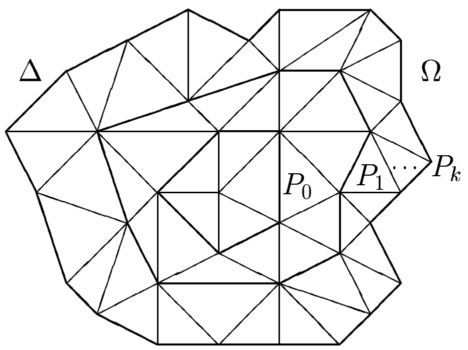

Fig. 1. Triangulation Δ (nested polygons).

§2. Main Results

Let Δ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$. Given an integer $q \geq 2$, we denote by $S_q^1(\Delta) = \{s \in C^1(\Omega) : s|_T \in \Pi_q \;\; for\; all \;\; T \in \Delta\}$ the space of bivariate splines of degree q and smoothness one (with respect to Δ). Here $\Pi_q = \text{span}\{x^\alpha y^\beta : \alpha, \beta \geq 0,\; \alpha+\beta \leq q\}$ denotes the space of bivariate polynomials of total degree q. We investigate the following problem. Construct sets $\{z_1, \ldots, z_d\}$ in Ω, where $d = \dim S_q^1(\Delta)$, such that for each function $f \in C(T)$, a unique spline $s \in S_q^1(\Delta)$ exists which satisfies the Lagrange interpolation conditions $s(z_\nu) = f(z_\nu), \nu = 1, \ldots, d$. If we consider not only function values of f but also partial derivatives, then we speak of Hermite interpolation conditions.

The Class of Triangulations. We consider the following general type of triangulations Δ. The vertices of Δ are the vertices of closed simple polygons $P_0, P_1, \ldots, P_k$ which are nested and one vertex inside P_0. This means that $\Omega_{\mu-1} \subset \Omega_\mu$, where Ω_μ is the closed (not necessarily convex) polyhedron with boundary $P_\mu, \mu = 0, \ldots, k$, and Δ is a triangulation of $\Omega := \Omega_k$ (see Figure 1). To be more precise, we note that the vertices of P_μ are connected by line segments with the vertices of $P_{\mu+1}, \mu = 0, \ldots, k-1$. On the other hand, for each closed simple polygon P_μ, there is no additional line segment connecting two vertices of $P_\mu, \mu = 0, \ldots, k$. In order to construct interpolation points for $S_3^1(\Delta)$, we assume that the triangulation Δ has the following properties:
($T1$) Each vertex of P_μ is connected with at least two vertices of $P_{\mu+1}, \mu = 0, \ldots, k-1$.
($T2$) There exist vertices w_μ of $P_\mu, \mu = 0, \ldots, k$, such that w_μ and $w_{\mu+1}$ are connected, and each vertex w_μ is connected with at least three vertices of $P_{\mu+1}, \mu = 0, \ldots, k-1$.

Remark 1. (i) Since the polygons P_μ grow with increasing index μ, it is natural to assume that the number of vertices of $P_{\mu+1}$ is greater than the number of vertices of $P_\mu, \mu = 0, \ldots, k-1$. Then it is natural to connect the vertices of the polygons in such a way that the properties ($T1$) and ($T2$) are satisfied. (ii) Moreover, the properties ($T1$) and ($T2$) of Δ remain valid if Δ is deformed, i.e., the location of the vertices of Δ are changed but the

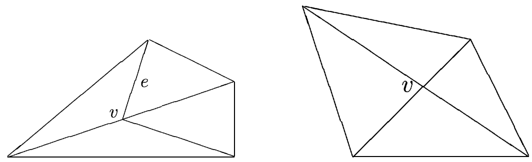

Fig. 2. Degenerate edge, respectively singular vertex.

connection of the vertices remain unchanged. (In other words, the graphs of the triangulation Δ and the deformed triangulation are the same.)

Decomposition of the Domain. In order to construct interpolation points, we decompose the domain Ω into finitely many sets $V_0 \subset V_1 \subset \cdots \subset V_m = \Omega$, where each set V_i is the union of closed triangles of $\Delta, i = 0, \ldots, m$. Let V_0 be an arbitrary closed triangle of Δ in Ω_0. We define the sets $V_1 \subset \cdots \subset V_m$ by induction according to the following rule: If V_{i-1} is defined, then we choose a vertex v_i of Δ with the following property: Let $T_{i,1}, \ldots, T_{i,n_i} (n_i \geq 1)$ be all the triangles of Δ with vertex v_i having a common edge with V_{i-1}. (Since Δ satisfies property $(T1)$, we have $n_i \leq 2$.) We set $V_i = V_{i-1} \cup \overline{T}_{i,1} \cup \ldots \cup \overline{T}_{i,n_i}$. (Note that we choose the vertex v_i in such a way that at least one such triangle exists.)

The vertices $v_i, i = 1, \ldots, m$, are chosen as follows. After choosing V_0 to be an arbitrary closed triangle of Δ in Ω_0, we pass through the vertices of P_0 in clockwise order by applying the above rule. (It is clear that the choice of these vertices is unique.) Now, we assume that we have passed through the vertices of $P_{\mu-1}$. Then w.r.t. clockwise order, we choose the first vertex of P_μ greater than w_μ which is connected with at least two vertices of $P_{\mu-1}$. Then we pass through the vertices of P_μ in clockwise order until w_μ^- and pass through the vertices of P_μ in counterclockwise order until w_μ^+ by applying the above rule. (Here w_μ^+ denotes the vertex next to w_μ in clockwise order and w_μ^- denotes the vertex next to w_μ in counterclockwise order.) Finally, we choose the vertex w_μ. (It is clear that the choice of the vertices is unique.) In this way, we obtain the sets $V_0 \subset V_1 \subset \ldots \subset V_m = \Omega$.

Construction of Interpolation Sets. The choice of interpolation points depends on the following properties of the triangulation Δ.

Definition 2. (i) *An interior edge e with vertex v of the triangulation Δ is called* degenerate *at v if the edges with vertex v adjacent to e lie on a line.* (ii) *An interior vertex v of Δ is called* singular *if v is a vertex of exactly four edges and these edges lie on two lines.* (iii) *An interior vertex v of Δ on the boundary of a given subtriangulation Δ' of Δ is called* semi-singular of type 1 *w.r.t. Δ' if exactly one edge with endpoint v is not contained in Δ' and this edge is degenerate at v.* (iv) *An interior vertex v of Δ on the boundary of a given subtriangulation Δ' of Δ is called* semi-singular of type 2 *w.r.t. Δ' if exactly two edges with endpoint v are not contained in Δ' and these edges are degenerate at v.* (v) *A vertex v of Δ is called* semi-singular *w. r. t. Δ' if v satisfies (iii) or (iv).*

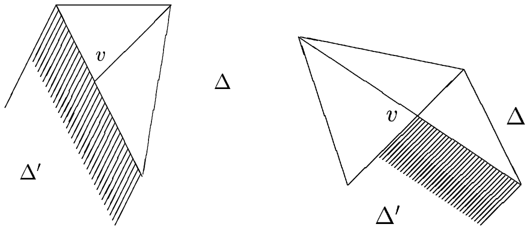

Fig. 3. Semi-singular vertex.

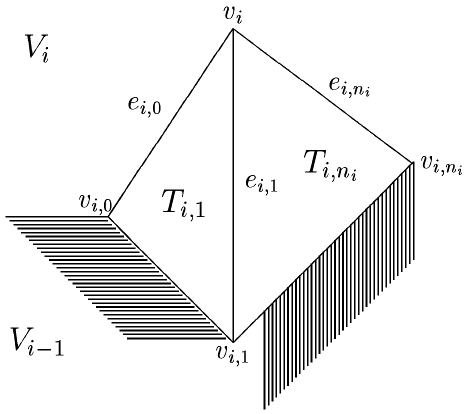

Fig. 4. The set $V_i \setminus V_{i-1}$.

Now, we construct interpolation sets for $S_3^1(\Delta)$ inductively as follows. First, we choose interpolation points on V_0 and then on $V_i \setminus V_{i-1}, i = 1, \dots, m$. In the first step, we choose 10 different points (respectively 10 Hermite interpolation conditions) on V_0 which admit unique Lagrange interpolation by the space Π_3. (For example, we may choose four parallel line segments l_ν in V_0 and ν different points on each $l_\nu, \nu = 1, 2, 3, 4$.)

Now, we assume that we have already chosen interpolation points on V_{i-1}. Then we choose interpolation points on $V_i \setminus V_{i-1}$ as follows. By the above decomposition of Ω, $V_i \setminus V_{i-1}$ is the union of consecutive triangles $T_{i,1}, \dots, T_{i,n_i}$ with vertex v_i having common edges with V_{i-1}. We denote the consecutive endpoints of these edges by $v_{i,0}, v_{i,1}, \dots, v_{i,n_i}$. Moreover, the edges $[v_{i,j}, v_i]$ are denoted by $e_{i,j}, j = 0, \dots, n_i$ (see Figure 4).

The choice of interpolation points on $V_i \setminus V_{i-1}$ depends on the following properties of the subtriangulation $\Delta_i = \{T \in \Delta : T \subset V_i\}$ at the vertices $v_{i,0}, \dots, v_{i,n_i}$: (i) $e_{i,j}$ is non-degenerate at $v_{i,j}$. (ii) $e_{i,j}$ is non-degenerate at $v_{i,j}$ and in addition, $v_{i,j}$ is semi-singular w.r.t. Δ_i.

For $j \in \{0, n_i\}$, we set $c_{i,j} = 1$ if (ii) holds; and $c_{i,j} = 0$ otherwise. For $0 < j < n_i$, we set $c_{i,j} = 1$ if (i) holds; and $c_{i,j} = 0$ otherwise. Moreover, we set $c_i = \sum_{j=0}^{n_i} c_{i,j}$ and note that $0 \le c_i \le 3$. For Lagrange interpolation, we choose the following points on $V_i \setminus V_{i-1}$: If $c_i = 3$, then no point is chosen. If $c_i = 2$, then we choose v_i. If $c_i = 1$, then we choose v_i and one additional point on some edge $e_{i,j}$ with $c_{i,j} = 0$. If $c_i = 0$, then we choose v_i and two additional points on two different edges. For Hermite interpolation, we require the following interpolation conditions for $s \in S_3^1(\Delta)$ at the vertex v_i: If $c_i = 3$, then no interpolation condition is required at v_i. If $c_i = 2$, then we require $s(v_i) = f(v_i)$. If $c_i = 1$, then we require $s(v_i) = f(v_i)$ and $\frac{\partial s}{\partial e_{i,j}}(v_i) = \frac{\partial f}{\partial e_{i,j}}(v_i)$, where $e_{i,j}$ is some edge with $c_{i,j} = 0$. If $c_i = 0$, then we require $s(v_i) = f(v_i)$, $\frac{\partial s}{\partial x}(v_i) = \frac{\partial f}{\partial x}(v_i)$ and $\frac{\partial s}{\partial y}(v_i) = \frac{\partial f}{\partial y}(v_i)$. (For simplicity, we have denoted the derivative in direction of a unit vector parallel to the edge $e_{i,j}$ by $\frac{\partial}{\partial e_{i,j}}$.) By the above construction, we obtain a set of points for Lagrange interpolation (respectively a set of Hermite interpolation conditions).

Theorem 3. *If the triangulation Δ satisfies the properties $(T1)$ and $(T2)$, then there exists a unique spline in $S_3^1(\Delta)$ which satisfies the above Lagrange (respectively Hermite) interpolation conditions. In particular, the total number of interpolation conditions is equal to the dimension of $S_3^1(\Delta)$.*

Proof: Let $s \in S_3^1(\Delta)$ be a given spline which satisfies the homogenous Lagrange (respectively Hermite) interpolation conditions. We will show that $s = 0$ on Ω and that the total number of interpolation conditions on Ω is equal to the dimension of $S_3^1(\Delta)$.

First, we show by induction that $s = 0$ on $V_i, i = 0, \dots, m$. It is clear that the interpolation conditions on V_0 imply $s = 0$ on V_0. Now, we assume that $s = 0$ on V_{i-1} for some $i \in \{1, \dots, m\}$ and consider V_i. Set $\tilde{e}_{i,j} = [v_{i,j-1}, v_{i,j}]$ and $p_{i,j} = s|_{T_{i,j}} \in \Pi_3, j = 1, \dots, n_i$. For simplicity, we omit the index i of $v_i, v_{i,j}, e_{i,j}, \tilde{e}_{i,j}, T_{i,j}, p_{i,j}$ and n_i. Since $s \in C^1(\Omega)$, it follows from the induction hypothesis (i.e., $s = 0$ on V_{i-1}) that for all $j \in \{1, \dots, n\}$,

$$\frac{\partial^{\alpha+\beta} p_j}{\partial^\alpha \tilde{e}_j \partial^\beta e_{j-1}} = 0 \quad \text{and} \quad \frac{\partial^{\alpha+\beta} p_j}{\partial^\alpha \tilde{e}_j \partial^\beta e_j} = 0 \quad \text{on } \tilde{e}_j \tag{1}$$

for all $\alpha \geq 0, \beta = 0, 1$ and $\alpha + \beta \leq 3$. We consider the following cases.

Case 1. $c_i = 0$. For Lagrange interpolation, we may assume that the three interpolation points are chosen on the edges e_0 and e_1. Since p_1 is zero at these points and p_1 satisfies the zero properties (1), it follows that $p_1 = 0$. The same arguments hold for Hermite interpolation. Since $s = 0$ on T_1 and $s \in C^1(\Omega)$, we obtain $p_2(v) = 0$, $\frac{\partial p_2}{\partial \tilde{e}_2}(v) = 0$, and $\frac{\partial p_2}{\partial e_2}(v) = 0$. This together with (1) for p_2 implies $p_2 = 0$, and therefore $s = 0$ on V_i.

Case 2. $c_i = 1$. First, we consider the case $c_{i,1} = 1$, where $n = 2$. In this case e_1 is non-degenerate at v_1. Hence, we have $e_1 = \gamma_1 \tilde{e}_1 + \gamma_2 \tilde{e}_2$, where $\gamma_1, \gamma_2 \in \mathbb{R} \setminus \{0\}$. Thus, $\frac{\partial^2 p_1}{\partial^2 e_1}(v_1) = \gamma_1 \frac{\partial^2 p_1}{\partial \tilde{e}_1 \partial e_1}(v_1) + \gamma_2 \frac{\partial^2 p_1}{\partial \tilde{e}_2 \partial e_1}(v_1)$. It follows from (1) that $\frac{\partial^2 p_1}{\partial^2 e_1}(v_1) = 0$. Since v is an interpolation point, we obtain that $p_1 = 0$ on e_1. Analogously as in Case 1, we conclude $s = 0$ on V_i.

Now, we consider $c_{i,0} = 1$, i.e., v_0 is semi-singular of type 2 w.r.t. Δ_i and e_0 is non-degenerate at v_0. Let e be the edge outside of Δ_i attached to v_0, which is not lying on the same line as e_0. Denote by $\tilde{T}_j, j = 1, \dots, 3$, the triangles with vertex v_0 outside of Δ_i in counterclockwise order and set $\tilde{p}_j = s|_{\tilde{T}_j} \in \Pi_3, j = 1, \dots, 3$. Since $s = 0$ on V_{i-1}, we have

$$\frac{\partial^2 p_1}{\partial e \partial e_0}(v_0) = \frac{\partial^2 \tilde{p}_1}{\partial e \partial e_0}(v_0) = -\frac{\partial^2 \tilde{p}_2}{\partial e \partial(-e_0)}(v_0) = \frac{\partial^2 \tilde{p}_3}{\partial(-e) \partial(-e_0)}(v_0) = 0 \ .$$

Since e_0 is non-degenerate at v_0, analogously as above, it follows from (1) that $\frac{\partial^2 p_1}{\partial^2 e_0}(v_0) = 0$. As in the above case, we conclude from the interpolation conditions that $s = 0$ on V_i.

Finally, we consider $c_{i,n} = 1$, i.e., v_n is semi-singular w.r.t. Δ_i and e_n is non-degenerate at v_n. We may assume that v_n is semi-singular of type 1 w.r.t. Δ_i, since the remaining case can be treated analogous to the above case $c_{i,0} = 1$. Let e be the edge outside of Δ_i attached to v_n, denote by $\tilde{T}_j, j = 1, 2$, the triangles with vertex v_n outside of Δ_i in clockwise order and set $\tilde{p}_j = s|_{\tilde{T}_j} \in \Pi_3, j = 1, 2$. Since $s = 0$ on V_{i-1}, we have

$$\frac{\partial^2 p_n}{\partial e \partial e_n}(v_n) = \frac{\partial^2 \tilde{p}_1}{\partial e \partial e_n}(v_n) = -\frac{\partial^2 \tilde{p}_2}{\partial e \partial (-e_n)}(v_n) = 0 \ .$$

Since e_n is non-degenerate at v_0, analogously as above, it follows from (1) that $\frac{\partial^2 p_n}{\partial^2 e_n}(v_n) = 0$. As in the above case, we conclude from the interpolation conditions that $s = 0$ on V_i.

Case 3. $c_i = 2$. Here, we have three cases which can be treated by analogous arguments as in Case 2.

Case 4. $c_i = 3$. By analogous arguments as in Case 2, we obtain $\frac{\partial^2 p_1}{\partial^2 e_0}(v_0) = 0$, $\frac{\partial^2 p_1}{\partial^2 e_1}(v_1) = 0$ and $\frac{\partial^2 p_2}{\partial^2 e_2}(v_2) = 0$. It is well known (cf. [4], p. 124) that each univariate polynomial p of degree 3 on an interval $[a, b]$ satisfies $6p(a) + 2(b - a)p'(a) = 6p(b) - 4(b - a)p'(b) + (b - a)^2 p''(b)$. It follows that

$$3p_1(v) + \alpha_j \frac{\partial p_1}{\partial e_j}(v) = 0, \qquad j = 0, \dots, 2, \tag{2}$$

where α_j is the length of $e_j, j = 0, \dots, 2$. If e_0 and e_2 lie on a line, then it is easy to see that these equations imply

$$p_1(v) = \frac{\partial p_1}{\partial e_0}(v) = \frac{\partial p_1}{\partial e_1}(v) = \frac{\partial p_1}{\partial e_2}(v) = 0. \tag{3}$$

If e_0 and e_2 do not lie on a line, then we have $\sin(\theta_1 + \theta_2)e_1 = \sin(\theta_2)e_0 + \sin(\theta_1)e_2$, where $\theta_j \in (0, \pi)$ is the angle in $T_j, j = 1, 2$ at v. Thus,

$$\sin(\theta_1 + \theta_2)\frac{\partial p_1}{\partial e_1}(v) = \sin(\theta_2)\frac{\partial p_1}{\partial e_0}(v) + \sin(\theta_1)\frac{\partial p_1}{\partial e_2}(v) \ .$$

This and (2) lead to a homogenous linear system with corresponding determinant $3(-\alpha_1\alpha_2 \sin(\theta_2) - \alpha_0\alpha_1 \sin(\theta_1) + \alpha_0\alpha_2 \sin(\theta_1 + \theta_2))$. It is obvious that for $\theta_1 + \theta_2 > \pi$, this determinant is nonzero. Moreover, this also holds for $\theta_1 + \theta_2 < \pi$, since the area of the triangle with vertices v, v_0, and v_2 is different from the sum of the areas of the triangles T_1 and T_2. It follows that (3) holds in all cases. Now, analogous to Case 1, we obtain $s = 0$ on V_i. This shows $s = 0$ on V_i.

Finally, we show that the total number $M(\Delta)$ of interpolation conditions for $S_3^1(\Delta)$ is equal to the dimension of $S_3^1(\Delta)$. It follows from the above proof that $\dim S_3^1(\Delta) \le M(\Delta)$. On the other hand, it was shown in [6] that $L(\Delta) = 3V_B(\Delta)+2V_I(\Delta)+\sigma(\Delta)+1 \le \dim S_3^1(\Delta)$, where $V_B(\Delta)$ (respectively $V_I(\Delta)$) is the number of boundary (respectively interior) vertices of Δ and $\sigma(\Delta)$ is the number of singular vertices of Δ. Thus, it remains to show $L(\Delta) = M(\Delta)$. We prove by induction that $L(\Delta_i) = M(\Delta_i), i = 0,\ldots,m$. Since we have chosen $\dim \Pi_3 = 10$ interpolation conditions on Δ_0, $L(\Delta_0) = M(\Delta_0)$. We assume $L(\Delta_{i-1}) = M(\Delta_{i-1})$ and show $L(\Delta_i) = M(\Delta_i)$. If $n_i = 1$, then $V_B(\Delta_i) = V_B(\Delta_{i-1})+1, V_I(\Delta_i) = V_I(\Delta_{i-1})$. Thus, $L(\Delta_i) = L(\Delta_{i-1})+3$. By the choice of interpolation conditions, we have $M(\Delta_i) = M(\Delta_{i-1})+3$. Thus, $L(\Delta_i) = M(\Delta_i)$. If $n_i = 2$, then $V_B(\Delta_i) = V_B(\Delta_{i-1})$, $V_I(\Delta_i) = V_I(\Delta_{i-1})+1$. Thus, $L(\Delta_i) = L(\Delta_{i-1})+2$. If $c_{i,1} = 1$, then by the choice of interpolation conditions, $M(\Delta_i) = M(\Delta_{i-1})+2$. Thus, $L(\Delta_i) = M(\Delta_i)$. Now, we consider $c_{i,1} = 0$. We assume that $v_{i,1}$ is not singular. (Note that by property (T1) of Δ, the vertex inside P_0 is the only vertex which may be singular.) We claim that there exist a unique integer $i_0 \le i-1$ and $j \in \{0,\ldots,n_{i_0}\}$ such that $v_{i,1} = v_{i_0}$ is semi-singular w.r.t Δ_{i_0} and $c_{i_0,j} = 1$. First, we note that it follows from properties (T1) and (T2) of Δ that $v_{i_1} := v_{i,1}$ is not semi-singular w.r.t. Δ_{i_1}. We consider two cases.

Case 1. Suppose the vertices $v_{i,1}$ and v_{i-2} are connected by a line segment e.

If the edges $e_{i,1}$ and e do not lie on a line, then $i_0 = i-1$ and $c_{i_0,1} = 1$. (In this case, $v_{i,1}$ is semi-singular of type 1 w.r.t Δ_{i-1}.) If the edges $e_{i,1}$ and e do lie on a line, then $i_0 = i-2$ and, since $v_{i,1}$ is not singular, $c_{i_0,n_{i_0}} = 1$. (In this case $v_{i,1}$ is semi-singular of type 2 w.r.t Δ_{i-2}.) Moreover, $v_{i,1}$ is not semi-singular w.r.t Δ_{i_2}, where $v_{i_2} := v_{i,2}$ since at least three edges of Δ_i outside of Δ_{i_2} are attached to $v_{i,1}$.

Case 2. Suppose the vertices $v_{i,1}$ and v_{i-2} are not connected by a line segment.

Let e be the edge which connects $v_{i,1}$ with the vertex on its polygon in counter-clockwise order. If the edges $e_{i,1}$ and e do not lie on a line, then we also have $i_0 = i-1$ and $c_{i_0,1} = 1$. Moreover, since $\tilde{e}_1$ is non-degenerate at $v_{i,1}$, it follows that $v_{i,1}$ is not semi-singular w.r.t Δ_{i_2}, where $v_{i_2} := v_{i,2}$. If the edges $e_{i,1}$ and e do lie on a line, then $v_{i,1}$ is semi-singular of type 2 w.r.t Δ_{i_2}, where $v_{i_2} := v_{i,2}$, and $c_{i_2,0} = 1$. Moreover, in this case, $v_{i,1}$ is semi-singular of type 1 w.r.t Δ_{i-1}, but $c_{i-1,1} = 0$. This shows that if $c_{i,1} = 0$ (and $v_{i,1}$ not singular), then there exists a unique integer $i_0 \le i-1$ such that $c_{i_0,j} = 1$. By the choice of interpolation conditions, it follows that $M(\Delta_i) = (M(\Delta_{i-1})-1)+3 = M(\Delta_{i-1})+2$. Thus, $L(\Delta_i) = M(\Delta_i)$. This proves Theorem 3. □

If a triangulation consists of subrectangles by adding one diagonal (of the same direction), then we speak of a Δ^1-partition. The next result on Δ^1-partitions which are deformed (see Remark 1) follows from Theorem 3.

Corollary 4. *Let Δ be a deformed Δ^1-partition. Then there exists a unique spline in $S_3^1(\Delta)$ which satisfies the Lagrange (respectively Hermite) interpola-*

tion conditions obtained by the above method.

Numerical examples. We interpolate Franke's test function (see [5] in this volume) by splines in $S_3^1(\Delta)$, where Ω is somewhat larger than $[0,1] \times [0,1]$ and Δ is a uniform triangulation of Ω consisting of nested polygons. By using 3747 (resp. 14403) interpolation points, we obtain an error of $1.61 * 10^{-4}$ (resp. $2.03 * 10^{-5}$) in the uniform norm. (In the case of non-uniform Δ, our method may be modified. If $V_i \setminus V_{i-1}$ is a convex (respectively non-convex) quadrangle with one diagonal, then the second diagonal is added (respectively one triangle of the two is subdivided into three subtriangles). In this case, the interpolation points are obtained easily by combining the methods in this paper and in [5].) The interpolating splines are computed by passing through the triangles and by solving several small systems instead of one large system.

Finally, we note that our basic principle of passing through the vertices of the nested polygons of Δ can also be applied to the space $S_q^1(\Delta), q \geq 4$, in combination with the algorithm for constructing interpolation points in [2]. Then, in contrast to [2], the choice of the vertices is unique.

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