Locally Linearly Independent Systems and Almost Interpolation

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Abstract. A simple method for constructing almost interpolation sets in the case of existence of locally linearly independent systems of basis functions is presented. Various examples of such systems, including translates of box splines and finite-element splines, are considered.

1. Introduction

In [16] we have shown how the well-known Schoenberg-Whitney condition for interpolation by univariate polynomial splines can be extended to multivariate splines or even to the general setting of real functions defined on a topological space. For this case it characterizes *almost interpolation sets* (AI*–sets*); *i.e.,* those configurations T such that in every neighborhood of T there exists a configuration $\tilde{T}$ (I*–set*) which admits Lagrange interpolation.

In practice it is clearly quite important to have algorithms of constructing I-sets. For instance, for a system $\{B_1, \ldots, B_n\}$ of univariate polynomial B-splines it is well-known that any set $T = \{t_1, \ldots, t_n\}$ such that $t_i \in \{t \in \mathbb{R} : B_i(t) \neq 0\}$, $i = 1, \ldots, n$ is an I-set w.r.t. span $\{B_1, \ldots, B_n\}$ (*support property*).

Since general methods of transforming AI-sets into an I-sets are available (see [16, Section 5]), and, on the other hand, no simple characterization of I-sets seems possible in the multivariate case, it would be desirable to find simple construction methods for AI-sets. In the above example of univariate polynomial B-splines, it can be easily seen that AI-sets are characterized by the condition

$$t_i \in \operatorname{supp} B_i := \overline{\{t \in \mathbb{R} : B_i(t) \neq 0\}}, \quad i = 1, \ldots, n\,.$$

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A certain extension of this *weak support property* to multivariate splines has been found in [25] (see also [16, Theorem 3.7 and Theorem 4.12]). However, it substantially differs from its univariate source. The disadvantage is that *each* basis of a multivariate spline space U has to be examined in order to check whether a configuration T is an AI–set.

Fortunately, this drawback can be overcome if U admits a *locally linearly independent basis* (*LI–basis*). Local linear independence was considered by de Boor and Höllig [6], Dahmen and Micchelli [13] and Jia [19] as a property of the integer translates of a box spline, and further investigated in [2,8,9,14,15,17,20,26]. (Particularly, Carnicer and Peña [9] have shown that I-sets with respect to a finite-dimensional space spanned by a locally linearly independent weak Descartes system of univariate continuous functions can be characterized by the support property.) The importance of this notion for the study of AI-sets follows from the fact that for a space U spanned by an LI–basis $\{u_1, \ldots, u_n\}$, a set $T = \{t_1, \ldots, t_s\}$, $s \leq n$ is an AI-set w.r.t. U if and only if there exists some permutation σ of $\{1, \ldots, n\}$ such that $t_i \in \operatorname{supp} u_{\sigma(i)}$, $i = 1, \ldots, s$ (Theorem 2.3).

For constructing AI-sets we are therefore interested in spaces which admit LI-bases. In Section 3 we present various examples of such spaces, including univariate generalized splines, translates of box splines, tensor product splines, continuous multivariate splines on simplex partitions and finite-element bivariate splines.

In this paper we shall follow the notations of [16].

2. Locally Linearly Independent Systems

In this section we describe some properties of LI–bases and their relationship to the problem of constructing almost interpolation sets. Although we are mostly interested in finite systems of functions and finite-dimensional linear spaces spanned by them, the theory of locally linear independence can be developed for certain infinite systems.

Let K be a topological space. We say that a system of nonzero functions $\{u_i\}_{i \in I} \subset F(K)$, is *locally finite* if for any $t \in K$ there exists a neighborhood $B(t)$ such that the set

$$\{i \in I : B(t) \cap \ \operatorname{supp}\ u_i \neq \emptyset\}$$

is finite. Particularly, we can consider the infinite series

$$\sum_{i \in I} a_i u_i(x), \quad x \in K,$$

taking into account the fact that for each fixed $x \in K$ only a finite number of terms is nonzero.

It is also quite clear that the local dimension

$$\text{l-dim}_{K'} U := \inf \{\dim U|_B : \ K' \subset B, \ B \text{ open}\}$$

is finite when $U \subset F(K)$ denotes a linear space spanned by a locally finite system of functions and K' is a finite set. Particularly, $\varphi(t) := \text{l-dim}_t U := \text{l-dim}_{\{t\}} U$ is well-defined for such spaces.

Definition 2.1. *A locally finite system $\{u_i\}_{i\in I} \subset F(K)$ is said to be* locally linearly independent (LI-system) *if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B' such that $t \in B' \subset B(t)$ and the subsystem*

$$\{u_i : B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent. The linear span of an LI-system is called LI-space.

The next theorem gives some equivalent definitions of LI-systems.

Theorem 2.2. *[17] Let $\{u_i\}_{i\in I} \subset F(K)$ be a locally finite system of functions and let $U = \operatorname{span}\{u_i : i \in I\}$. The following conditions are equivalent.*

1) $\{u_i\}_{i\in I}$ *is a locally linearly independent system.*
2) $\operatorname{l-dim}_t U = \operatorname{card}\{i \in I : t \in \operatorname{supp} u_i\},\ t \in K$.
3) $\operatorname{l-dim}_{K'} U = \operatorname{card} I_{K'},\ K' \subset K$ *finite, where*

$$I_{K'} := \{i \in I : K' \cap \operatorname{supp} u_i \neq \emptyset\}.$$

4) $\dim U|_B = \operatorname{card} I_B,\ B$ *open,* *if I_B is finite.*
5) *Given any open $B \subset K$,*

$$\sum_{i\in I} a_i u_i(x) = 0,\ x \in B, \quad \textit{implies} \quad a_i = 0 \quad \textit{for all} \ \ i \in I_B\,.$$

6) $\operatorname{supp}(\sum_{i\in I} a_i u_i) = \bigcup_{\substack{i\in I\\ a_i\neq 0}} \operatorname{supp} u_i$.

We note that the equivalence of 5) and 6) has been shown by Carnicer and Peña [8, Proposition 3.2] for the case when $\{u_i\}_{i\in I}$ is a finite system of functions.

It turns out that the statement of [16, Theorem 4.12] can be substantially simplified in the case that U is a finite-dimensional LI-space. Thus, we obtain a characterization of almost interpolation sets w.r.t. such spaces through a support property similar to [16, Support property (2.2)].

Theorem 2.3. *[17] Let $\{u_1, \ldots, u_n\} \subset F(K)$ be a locally linearly independent system and $U = \operatorname{span}\{u_1, \ldots, u_n\}$. A finite set $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$, is an AI-set w.r.t. U if and only if there exists some permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \operatorname{supp} u_{\sigma(i)},\ i = 1, \ldots, s.$$

When a particular system of functions is to be checked whether it is an LI-system or not, it is often helpful to make use of the following theorem.

Theorem 2.4. *[17] Let $\{u_i : i \in I\} \subset F(K)$ be a locally finite system of functions, $U = \operatorname{span}\{u_i : i \in I\}$. Assume that*

$$\overline{\operatorname{int}\operatorname{supp} u_i} = \operatorname{supp} u_i, \ i \in I. \tag{2.1}$$

If $\{u_{i|G_U} : i \in I\}$ is an LI-system, where G_U denotes the set of all points of continuity of $\varphi(t) = \text{l-dim}_t\, U$, then $\{u_i : i \in I\}$ is also an LI-system.

Note that (2.1) holds for any system of *continuous* functions u_i, since in that case $\{t \in K : u_i(t) \neq 0\}$ is open and everywhere dense in $\operatorname{supp} u_i$.

Some further properties of LI-spaces can be found in [17].

3. Examples of LI-systems

In this section we shall present various examples of LI-systems of uni- and multivariate functions. In every particular case when the construction of the LI-system is given, it is quite easy to characterize almost interpolation sets with the help of Theorem 2.3.

3.1. Univariate Polynomial B-Splines and Generalized Splines

It is well known that for the n-dimensional space $S_m(\Delta)$ of polynomial spline functions of degree m with r fixed knots there exists a basis $\{B_1, \ldots, B_n\}$ of functions with minimal support, the so-called B-splines (see e.g. [22]). In view of the properties of these functions, it is obvious that $\mathcal{U} := \{B_1, \ldots, B_n\}$ forms an LI-system for $S_m(\Delta)$.

This system can even be extended by an infinite knot sequence $\{x_i\}_{i=-\infty}^{\infty}$ to a system $\tilde{\mathcal{U}} := \{B_i\}_{i=-\infty}^{\infty}$ of B-splines of degree m, which is also locally linearly independent. Moreover, $\tilde{\mathcal{U}}$ forms a *totally positive system*; i.e., if $j_1 < \ldots < j_m$ are any integers, then $\det(B_{j_i}(t_k)) \geq 0$ for all points $t_1 < \ldots < t_m$ in $\mathbb{R}$, and strict positivity holds if and only if $t_i \in \operatorname{int}\operatorname{supp} B_{j_i}, i = 1, \ldots, m$. That case clearly implies that $\{t_1, \ldots, t_m\}$ is an I-set w.r.t. $\operatorname{span}\{B_{j_1}, \ldots, B_{j_m}\}$.

Sommer and Strauss introduced in [24] a class of generalized spline spaces which retains most of the features of the polynomial splines and includes important subclasses of spline spaces such as polynomial splines, generalized Chebyshevian splines and subspaces of splines in tension. The main result of [24] consists in constructing a basis for the generalized spline space which forms a weak Descartes system, admits a recursion relation and, as it can be easily seen from [24, Theorem 2.1], is locally linearly independent.

Carnicer and Peña [9] showed that a finite system of continuous functions on a closed interval of the real line is a locally linearly independent weak Descartes system if and only if its collocation matrices are almost strictly totally positive.

3.2. Translates of Box Splines

Let X be an arbitrary set of vectors (not necessarily distinct) containing a basis for $\mathbb{R}^k$,

$$X = \{x^i\}_{i=1}^n \subset \mathbb{R}^k \setminus \{0\}, \text{ span } X = \mathbb{R}^k .$$

(X will also be used to denote a $k \times n$ matrix.) Moreover, let

$$X(t) := \sum_{i=1}^{n} t_i x^i, \quad \text{if } t = \{t_1, \ldots, t_n\} \in \mathbb{R}^n,$$

$$\mathbb{B}(X) := \{V \subset X : \text{ card } V = \text{dim span } V = k\}.$$

Definition 3.1. *The* box spline $B(\cdot|X)$ *is a function defined by the rule*

$$\int_{\mathbb{R}^k} f(x) B(x|X) dx = \int_{[0,1)^n} f(X(t)) dt \quad (f \in C_0(\mathbb{R}^k)),$$

where $[0,1)^n$ *denotes the halfopen unit n-cube.*

The box splines have some important properties, including local support and piecewise polynomial structure (see e.g. [3,11,7]).

Theorem 3.2. *The following statements are true.*

1) $\text{supp}\, B(\cdot|X) = X([0,1]^n)$.
2) $\int_{\mathbb{R}^k} B(x|X) dx = 1$.
3) $B(\cdot|X) \in C^{d(X)-1}(\mathbb{R}^k) \setminus C^{d(X)}(\mathbb{R}^k)$ *where* $d(X) := \min\{\text{card}\, Y : Y \subset X,\ \text{span} X \setminus Y \neq \mathbb{R}^k\} - 1$.
4) *Set*

$$B_X := \{\sum_{j=1}^{k-1} c_j x^{i_j} + \sum_{j \in I'} b_j x^{i'_j} : 0 \le c_j \le 1,\ b_j = \pm 1,$$
$$1 \le i_1 < \ldots < i_{k-1} \le n\}$$

where $I' = \{i'_j\}$ *denotes the complementary set of* $\{i_j\}_{j=1}^{k-1}$ *w.r.t.* $\{1, \ldots, n\}$*. Then the restriction of* $B(\cdot|X)$ *to each component of the complement of* B_X *is a polynomial of total degree* $n-k$ *(*B_X *is called the* grid partition *of* $B(\cdot|X)$*).*

Now we consider the space $S(X)$ spanned by the integer translates of the box spline $B(\cdot|X)$,

$$S(X) := \text{ span } \{B(\cdot - \alpha|X) : \alpha \in \mathbb{Z}^k\}.$$

To determine AI-sets w.r.t. finite-dimensional subspaces of $S(X)$ we are interested in the question of whether the system of translates

$$\{B(\cdot - \alpha|X) : \alpha \in \mathbb{Z}^k\}$$

represents an LI-system. In fact, the following characterization due to de Boor and Höllig [5,6], Dahmen and Micchelli [12,13] and Jia [18,19] is true.

Theorem 3.3. *Let* $X = \{x^i\}_{i=1}^n \subset \mathbb{Z}^k \setminus \{0\}$ *with* $\operatorname{span} X = \mathbb{R}^k$. *The following conditions are equivalent.*

1) X *is* unimodular; *i.e.,*

$$|\det V| = 1 \text{ for all } V \subset \mathbb{B}(X).$$

2) $\{B(\cdot - \alpha|X) : \alpha \in \mathbb{Z}^k\}$ *is an* LI*-system.*
3) $\{B(\cdot - \alpha|X) : \alpha \in \mathbb{Z}^k\}$ *is a globally linearly independent system of functions.*

We note that several characterizations of unimodular matrices are available. (See, for example, [21, Chapters 19–21] and, in the case $k = 2$, [7, (II.29), p. 41].)

Let us now consider the case $k = 2$ more detailed. Suppose that the matrix $X \subset \mathbb{Z}^2$ contains the unit vectors e^1 and e^2. Following the notation of [11] we set

$$B_{tuvw}(\cdot) = B(\cdot|X_{tuvw})$$

with

$$X_{tuvw} = \{\underbrace{e^1, \ldots, e^1}_{t}, \underbrace{e^2, \ldots, e^2}_{u}, \underbrace{e^1 + e^2, \ldots, e^1 + e^2}_{v}, \underbrace{e^2 - e^1, \ldots, e^2 - e^1}_{w}\},$$

t, u, v, w nonnegative integers, $t, u \geq 1$. It follows easily from Theorem 3.2 that the grid partition w.r.t. B_{tuvw} is

1) a rectangular partition if $v = w = 0$ and $t, u \in \mathbb{N}$;
2) a type-1 triangulation if $w = 0$ and $t, u, v \in \mathbb{N}$ or $v = 0$ and $t, u, w \in \mathbb{N}$;
3) a type-2 triangulation if $t, u, v, w \in \mathbb{N}$.

Moreover, the following result is an immediate consequence of Theorem 3.3 and [7, (II.29)].

Proposition 3.4. *Assume that* $t, u \geq 1$. *Then the following statements hold.*

1) X_{tuvw} *is unimodular if and only if* $v = 0$ *or* $w = 0$.
2) *The system of translates*

$$\{B_{tuvw}(\cdot - \alpha_1, \cdot - \alpha_2) : \alpha_1, \alpha_2 \in \mathbb{Z}\}$$

is locally linearly independent if and only if $v = 0$ *or* $w = 0$.

3) *Each function* $B_{tuvw}(\cdot - \alpha_1, \cdot - \alpha_2)$ *is a bivariate spline function of total degree* $t + u + v + w - 2$ *and smoothness* $t + u + v + w - \max\{t, u, v, w\} - 2$.

Example 3.5. We consider two special cases.

1) Let $t = 2, u = 1, v = 2, w = 0$. Then the resulting system of translates of the box spline B_{2120} is an LI-system. The support of B_{2120} is given by the first figure in Fig. 3.1.
2) Let $t = u = v = w = 1$. Then the resulting system of translates fails to be an LI-system. The support of the corresponding box spline B_{1111} is given by the second figure in Fig. 3.1.

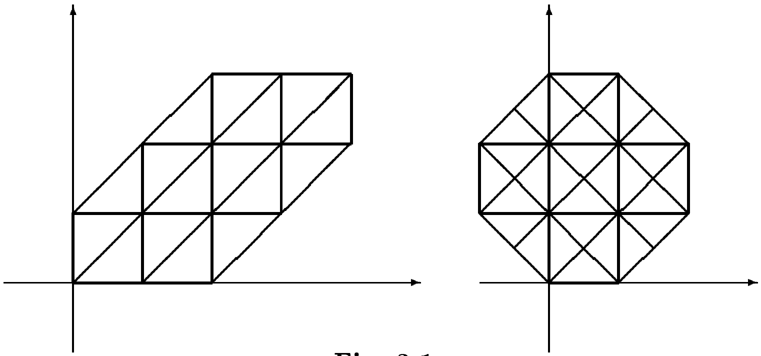

Fig. 3.1.

3.3. Tensor-Product Spaces

Let K_1 and K_2, respectively be topological spaces. Assume that $\mathcal{B}_1 = \{B_i\}_{i\in I_1}$ and $\mathcal{B}_2 = \{\tilde{B}_i\}_{i\in I_2}$ denote LI-systems in $F(K_1)$ and $F(K_2)$, respectively where I_1 and I_2 are index sets. Set

$$K := K_1 \times K_2,$$

the topological product of K_1 and K_2, and

$$\mathcal{B} := \{B_i\tilde{B}_j\}_{i\in I_1\, j\in I_2}.$$

Theorem 3.6. *The system $\mathcal{B}$ is an LI-system in $F(K)$.*

Proof: Assume that $\mathcal{B}$ fails to be an LI-system. Then there exist $t \in K$ and an open neighborhood M of t such that for some $(\tilde{i}, \tilde{j}) \in I_1 \times I_2$,

$$B_{\tilde{i}}\tilde{B}_{\tilde{j}} = \sum_{i\in\tilde{I}_1}\sum_{j\in\tilde{I}_2} \alpha_{ij} B_i \tilde{B}_j \ \text{ on } \ M,$$

where $\tilde{I}_1$ and $\tilde{I}_2$ are finite index sets such that $(\tilde{I}_1 \times \tilde{I}_2) \cup \{\tilde{i}, \tilde{j}\} = \{(i,j) \in I_1 \times I_2 : M \cap \text{ supp } B_i\tilde{B}_j \neq \emptyset\}$.

Without loss of generality assume that $M = M_1 \times M_2$ for some open $M_i \subset K_i$, $i = 1, 2$. Let $\tilde{t} = (\tilde{t}_1, \tilde{t}_2) \in M$ such that $\tilde{t}_i \in K_i$, $i = 1, 2$ and $\tilde{B}_{\tilde{j}}(\tilde{t}_2) \neq 0$. Then

$$B_{\tilde{i}}(\tilde{t}_1) = \sum_{i\in\tilde{I}_1}\left(\sum_{j\in\tilde{I}_2} \alpha_{ij} B_j(\tilde{t}_2)/\tilde{B}_{\tilde{j}}(\tilde{t}_2)\right) B_i(\tilde{t}_1).$$

This is even true for any $\tilde{t}_1 \in M_1$, contradicting the properties of the LI-system $\mathcal{B}_1$. ■

Remark 3.7. As an important application of the preceding theorem let us consider the case when $K_1 = [a, b] \subset \mathbb{R}$, $K_2 = [c, d] \subset \mathbb{R}$, and Δ and $\tilde{\Delta}$ are some knot partitions of K_1 and K_2, respectively. Assume that $S_m(\Delta)$ and $S_l(\tilde{\Delta})$ denote the linear spaces of polynomial splines on K_1 and K_2, respectively as has been defined in [16, Section 2]. By Section 3.1 the B-spline bases of $S_m(\Delta)$ and $S_l(\tilde{\Delta})$, denoted by

$$\mathcal{B}_1 = \{B_1, \dots, B_n\} \text{ and } \mathcal{B}_2 = \{\tilde{B}_1, \dots, \tilde{B}_{\tilde{n}}\},$$

respectively are LI-systems in $F(K_1)$ and $F(K_2)$, respectively. Hence by Theorem 3.6 the tensor-product space of bivariate polynomial spline functions of degree m in the first variable and degree l in the second one,

$$\mathcal{U} := \operatorname{span} \{B_i \tilde{B}_j\}_{i=1}^{n} {}_{j=1}^{\tilde{n}}$$

has a locally linearly independent basis, $\{B_i \tilde{B}_j\}_{i=1}^{n} {}_{j=1}^{\tilde{n}}$.

3.4. Multivariate Splines on Simplex Partitions

Let $x^0, \dots, x^k \in \mathbb{R}^k$, $k \geq 1$. The simplex

$$[x^0, \dots, x^k] := \{\sum_{i=0}^{k} \lambda_i x^i : \sum_{i=0}^{k} \lambda_i = 1,\ \lambda_i \geq 0\}$$

with vertices $x^0, \dots, x^k$ is called a *k-simplex*, if its volume in $\mathbb{R}^k$ is nonzero. Let

$$\Delta = \{S_i\}_{i \in I},$$

a family of finitely many k-simplices in $\mathbb{R}^k$ which satisfy the following property: If $S_i, S_j \in \Delta$, then $S_i \cap S_j$ is empty or a common face. Set

$$K = \bigcup_{i \in I} S_i.$$

For given integers r and d $(0 \leq r < d)$ we consider

$$\begin{aligned} S_d^r(\Delta) :=\{s \in C^r(K) : {} & s \text{ restricted to each } k\text{-simplex} \\ & \text{is a polynomial of total degree } d\}, \end{aligned}$$

the space of *polynomial splines of degree d and smoothness r on* Δ.

Suppose now that $\tilde{S} := [x^0, \dots, x^k]$ is a k-simplex in Δ. Then any $x = (x_1, \dots, x_k) \in \mathbb{R}^k$ can be identified by the *barycentric coordinates* $(\lambda_0, \dots, \lambda_k)$ w.r.t. $\tilde{S}$, where

$$\lambda_i = \lambda_i(x) = \frac{\operatorname{vol}_k[x^0, \dots, x^{i-1}, x, x^{i+1}, \dots, x^k]}{\operatorname{vol}_k[x^0, \dots, x^k]}, \quad i = 0, \dots, k.$$

Hence, λ_i is a linear polynomial in x. For any $\beta = (\beta_0, \ldots, \beta_k) \in \mathbb{Z}_+^{k+1}$, as usual we set

$$\lambda^\beta = \lambda_0^{\beta_0} \ldots \lambda_k^{\beta_k}, \; \beta! = \beta_0! \ldots \beta_k!, \; |\beta| = \beta_0 + \ldots + \beta_k.$$

Then

$$\varphi_\beta^d(\lambda) = \frac{d!}{\beta!}\lambda^\beta, \quad |\beta| = d$$

is a polynomial of degree d, and the set of all such functions $\{\varphi_\beta^d(\lambda) : |\beta| = d\}$ is a basis of π_d^k, the space of all polynomials of total degree at most d with k variables. Let $s \in S_d^r$ and $x \in \tilde{S}$. Then $s|_{\tilde{S}} \in \pi_d^k$ and

$$s(x) = s|_{\tilde{S}}(x) = \sum_{|\beta|=d} a_\beta^d(\tilde{S})\varphi_\beta^d(\lambda)$$

which is called the *Bézier-Bernstein* form of s w.r.t. $\tilde{S}$ (see [4]).

In addition, the set $\{(P_\beta(\tilde{S}), a_\beta^d(\tilde{S})) : |\beta| = d\}$ is called the *Bézier-net* of s w.r.t. $\tilde{S}$ where

$$P_\beta(\tilde{S}) = \sum_{i=0}^{d} \frac{\beta_i}{d}x^i,$$

the *domain point* and each $a_\beta^d(\tilde{S})$ is called *Bézier-ordinate*, associated with $P_\beta(\tilde{S})$.

The Case $r = 0$

Let $T_l = [x^0, \ldots, x^l]$ be an l-simplex in $\mathbb{R}^k$ where $0 \le l < k$ and let

$$S_1 = [x^0, \ldots, x^l, x^{l+1}, \ldots, x^k] \in \Delta,$$

$$S_2 = [x^0, \ldots, x^l, y^{l+1}, \ldots, y^k] \in \Delta$$

be two adjacent k-simplices with $S_1 \cap S_2 = T_l$. Suppose that $s \in S_d^0(\Delta)$. Then

$$s|_{S_1}(x) = \sum_{|\beta|=d} a_\beta^d(S_1)\varphi_\beta^d(\lambda), \quad x \in S_1,$$

$$s|_{S_2}(x) = \sum_{|\beta|=d} a_\beta^d(S_2)\varphi_\beta^d(\eta), \quad x \in S_2,$$

where $\lambda = (\lambda_0, \ldots, \lambda_k)$ and $\eta = (\eta_0, \ldots, \eta_k)$ are the barycentric coordinates w.r.t. S_1 and S_2, respectively. Clearly, $x \in T_l$ if and only if

$$x = \sum_{i=0}^{k} \lambda_i x^i = \sum_{i=0}^{k} \eta_i x^i$$

which implies that $\lambda_i = \eta_i, i = 0, \ldots, k$. Moreover, it follows that if $s|_{S_1 \cup S_2} \in C(S_1 \cup S_2)$,

$$a_\beta^d(S_1) = a_\beta^d(S_2)$$

for all $\beta = (\beta_0, \ldots, \beta_l, 0, \ldots, 0)$ with $\beta_0 + \ldots + \beta_l = d$.

Let $\mathcal{B}_d(\Delta)$ denote the set of all domain points of all k-simplices $S_i, i \in I$. Given a point $P = P_\beta(\tilde{S}) \in \mathcal{B}_d(\Delta)$, let λ_P be the linear functional defined on $S_d^0(\Delta)$ with the property that

$$\lambda_P s = a_\beta^d(\tilde{S}),$$

the Bézier-ordinate of s w.r.t. $\tilde{S}$ (associated with P).

The following dimension formula and explicit construction of a basis for $S_d^0(\Delta)$ are due to Alfeld, Schumaker and Sirvent [1].

Theorem 3.8. *The following statements hold.*

1) $\dim S_d^0(\Delta) = \operatorname{card} \mathcal{B}_d(\Delta)$.
2) *There exists a basis of* $S_d^0(\Delta)$ *given by*

$$\mathcal{B} := \{B_P \in S_d^0(\Delta) : \lambda_Q B_P = \delta_{QP} \quad \text{for all} \quad P, Q \in \mathcal{B}_d(\Delta)\}.$$

We are able to show that $\mathcal{B}$ is locally linearly independent.

Theorem 3.9. *The basis* $\mathcal{B}$ *of* $S_d^0(\Delta)$ *defined above is an LI-system.*

Proof: Assume that $\mathcal{B}$ fails to be locally linearly independent. Then we can find a k-simplex $\tilde{S}$ such that for some points $P_\mu, P_\rho \in \mathcal{B}_d(\Delta)$,

$$\mathcal{B}_{P_\mu} = \sum_\rho d_\rho \mathcal{B}_{P_\rho}$$

on $\tilde{S}$ where we clearly assume that $\tilde{S} \subset \operatorname{supp} \mathcal{B}_{P_\mu}, \tilde{S} \subset \operatorname{supp} \mathcal{B}_{P_\rho}, d_\rho \neq 0$ and $P_\mu \neq P_\rho$, all P_ρ. It is obvious that $P_\mu \in \tilde{S}$, because otherwise for the Bézier-ordinates $a_\beta^d(\tilde{S}; \mathcal{B}_{P_\mu})$ of $\mathcal{B}_{P_\mu}$,

$$0 = \lambda_Q \mathcal{B}_{P_\mu} = a_\beta^d(\tilde{S}; \mathcal{B}_{P_\mu}),$$

for all $Q = Q_\beta(\tilde{S}) \in \mathcal{B}_d(\Delta) \cap \tilde{S}$. This would imply that

$$\mathcal{B}_{P_\mu} \equiv 0 \quad \text{on} \quad \tilde{S},$$

a contradiction. Analogously we have that $P_\rho \in \tilde{S}$ for all ρ. Thus $P_\mu = P_{\tilde{\beta}}(\tilde{S})$ and

$$a_{\tilde{\beta}}^d(\tilde{S}; \mathcal{B}_{P_\mu}) = 1, \quad a_{\tilde{\beta}}^d(\tilde{S}; \mathcal{B}_{P_\rho}) = 0$$

where $a^d_{\tilde{\beta}}(\tilde{S};\mathcal{B}_{P_\rho})$ denotes the Bézier-ordinate of $\mathcal{B}_{P_\rho}$, associated with P_μ. Comparing now the coefficients of $\mathcal{B}_{P_\mu}$ and $\sum_\rho d_\rho \mathcal{B}_{P_\rho}$ on $\tilde{S}$, we obtain

$$1 = a^d_{\tilde{\beta}}(\tilde{S};\mathcal{B}_{P_\mu}) = \sum_\rho d_\rho a^d_{\tilde{\beta}}(\tilde{S};\mathcal{B}_{P_\rho}) = 0,$$

a contradiction. ■

Example 3.10. Let $k = 2$ and assume that K is a simply connected polygonal domain in $\mathbb{R}^2$. Moreover, let Δ be a regular triangulation of K (see [16, Example 3.2]). Then by the above arguments the space $S^0_d(\Delta)$ of continuous spline functions of degree d admits an LI-basis $\mathcal{B} = \{B_P\}_{P\in\mathcal{B}_d(\Delta)}$ where B_P is defined as above using the Bézier-Bernstein form.

3.5 Finite-Element Bivariate Splines

Let $\Delta = \{\Delta_i : i = 1,\ldots,N\}$ be a triangulation of a domain $K \subset \mathbb{R}^2$. Suppose that the vertices and the edges of Δ are denoted by $v_1,\ldots,v_V$ and $\epsilon_1,\ldots,\epsilon_E$, respectively. Following the notation of [23] we define the space of *super splines of degree d and smoothness r,ρ*, with $r \le \rho < d$, by

$$S^{r,\rho}_d(\Delta) = \{s \in C^r(K) : s|_{\Delta_i} \in \pi_d,\ i = 1,\ldots,N,$$
$$s \in C^\rho(v_j),\ j = 1,\ldots,V\},$$

where π_d is the space of bivariate polynomials of total degree d, and $C^\rho(v_j)$ denotes the set of functions which are ρ times continuously differentiable at the point v_j.

Suppose that $0 \le 2r \le \rho$ and $d \ge 2\rho + 1$. In this case it has been shown by Schumaker [23, Section 4] that a basis for $S^{r,\rho}_d(\Delta)$ can be constructed by using the classical finite-element method. We describe this basis as follows. For every vertex v_i, $i = 1,\ldots,V$ consider Hermite interpolation conditions,

$$\frac{\partial^{\mu+\nu} s}{\partial x^\mu \partial y^\nu}(v_i) = a^i_{\mu,\nu}, \quad 0 \le \mu+\nu \le \rho. \tag{3.4}$$

On each edge ϵ_i, $i = 1,\ldots,E$ choose points $\xi^{i,\nu}_1,\ldots,\xi^{i,\nu}_{d-1-2\rho+\nu} \in \epsilon_i$, $\nu = 0,\ldots,r$ and consider the following interpolation conditions,

$$\frac{\partial^\nu s}{\partial n^\nu_i}(\xi^{i,\nu}_l) = a^i_{\nu,l},\ \nu = 0,\ldots,r,\ l = 1,\ldots,d-1-2\rho+\nu, \tag{3.5}$$

where $\frac{\partial^\nu}{\partial n^\nu_i}$ denotes the ν-th order normal derivative w.r.t. the edge ϵ_i. Finally, take $N_{d,r,\rho} := \binom{d-3r-1}{2} - 3\binom{p-2r}{2}$ points $\zeta^i_1,\ldots,\zeta^i_{N_{d,r,\rho}}$ inside each triangle Δ_i in such a way that Lagrange interpolation conditions

$$s(\zeta^i_l) = a^i_l, \quad l = 1,\ldots,N_{d,r,\rho} \tag{3.6}$$

together with the conditions at the vertices and on the edges of the triangle Δ_i uniquely determine a polynomial of total degree d on that triangle. See [23, Theorem 4.1 and Remark 4] for the details concerning existence and construction of the points ζ_l^i (so-called "type-2 data").

In view of a theorem by Ženíšek [27], the piecewise polynomial function defined an each triangle Δ_i by conditions (3.4)–(3.6) necessarily belongs to $S_d^{r,\rho}(\Delta)$. Since the total number of conditions equals $n := \dim S_d^{r,\rho}(\Delta)$, we conclude that (3.4)-(3.6) is a well-posed Hermite interpolation problem for $S_d^{r,\rho}(\Delta)$. Therefore, the corresponding fundamental functions $s_1, \ldots, s_n$ form a basis for $S_d^{r,\rho}(\Delta)$. (Recall that the *fundamental functions* $u_1, \ldots, u_n$ of an interpolation problem $F_i(u) = a_i$, $i = 1, \ldots, n$, $u \in U$, $\dim U = n$, are determined by the conditions $F_i(u_j) = \delta_{ij}$ where F_i, $i = 1, \ldots, n$ are certain functionals and $\delta_{ii} = 1$, $\delta_{ij} = 0$ if $i \neq j$.)

Theorem 3.11. *The fundamental functions $s_1, \ldots, s_n$ of the interpolation problem (3.4)–(3.6) form a locally linearly independent basis for the space of super splines $S_d^{r,\rho}(\Delta)$, $0 \leq 2r \leq \rho$, $d \geq 2\rho + 1$.*

Proof: Consider first the supports of s_j, $j = 1, \ldots, n$. If s_j corresponds to a point v_i, $\xi_l^{i,\nu}$ or ζ_l^i, then supp s_j is evidently the union of all triangles with common vertex at v_i, the union of two triangles with common edge ϵ_i, or the triangle Δ_i, respectively. By Theorem 2.4 it suffice to check the locally linear independence of $\{s_j : j = 1 \ldots, n\}$ on the continuity set G_U of local dimension; *i.e.*, on the interiors of the triangles Δ_i. If now $x \in$ int Δ_{i_0}, then $x \in$ supp s_j exactly for those s_j which correspond to the points $v_i, \xi_l^{i,\nu}, \zeta_l^i$ lying on Δ_{i_0}. According to the construction of the conditions (3.4)–(3.6), the number of such s_j's equals the dimension of π_d and their restrictions to the triangle Δ_{i_0} are fundamental polynomials of a well-posed polynomial interpolation problem. Therefore, they are linearly independent in any neighborhood of x. This completes the proof. ■

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References

1. Alfeld P., Schumaker L. L. and Sirvent M., On dimension and existence of local bases for multivariate spline spaces, J. Approx. Theory **70** (1992), 243–264.
2. Ben-Artzi A. and Ron A., On the integer translates of a compactly supported functions: dual bases and linear projectors, SIAM J. Math. Anal. **21** (1990), 1550–1562.
3. Bojanov B. D., Hakopian H. A. and Sahakian A. A., *Spline Functions and Multivariate Interpolations*, Kluwer Academic Publishers, Dordrecht, 1993.
4. de Boor C., *B*-form basics, in *Geometric Modeling: Algorithms and New Trends* (Farin G. E., Ed.), SIAM, Philadelphia, 1987, 131–148.

5. de Boor C. and Höllig K., B-splines from parallelepipeds, J. Analyse Math. **42** (1982), 99–115.
6. de Boor C. and Höllig K., Bivariate box splines and smooth pp functions on a three direction mesh, J. Comput. Appl. Math. **9** (1983), 13–28.
7. de Boor C., Höllig K. and Riemenschneider S., *Box Splines*, Springer, New York, 1993.
8. Carnicer J. M. and Peña J. M., Least supported bases and local linear independence, Numer. Math. **67** (1994), 289–301.
9. Carnicer J. M. and Peña J. M., Spaces with almost strictly totally positive bases, Math. Nachrichten **169** (1994), 69–79.
10. Cheney E. W., *Multivariate Approximation Theory: Selected Topics*, CBMS–SIAM, Philadelphia, 1986.
11. Chui C. K., *Multivariate Splines*, CBMS–SIAM, Philadelphia, 1988.
12. Dahmen W. and Micchelli C. A., Translates of multivariate splines, Linear Algebra Appl. **52/53** (1983), 217–234.
13. Dahmen W. and Micchelli C. A., On the local linear independence of translates of a box spline, Studia Math. **82** (1985), 243–262.
14. Dahmen W. and Micchelli C. A., On multivariate E-splines, Advances in Math. **76** (1989), 33–93.
15. Davydov O. and Sommer M., Interpolation by weak Chebyshev spaces, preprint.
16. Davydov O., Sommer M. and Strauss H., On almost interpolation by multivariate splines, this volume.
17. Davydov O., Sommer M. and Strauss H., On almost interpolation and locally linearly independent bases, preprint.
18. Jia R.-Q., Linear independence of translates of a box spline, J. Approx. Theory **40** (1984), 158–160.
19. Jia R.-Q., Local linear independence of the translates of a box spline, Constr. Approx. **1** (1985), 175–182.
20. Ron A., Linear independence of the translates of an exponential box spline, Rocky Mountain J. Math. **22** (1992), 331–351.
21. Schrijver A., *Theory of linear and integer programming*, Wiley-Interscience, New York, 1986.
22. Schumaker L. L., *Spline Functions: Basic Theory*, Wiley-Interscience, New York, 1981.
23. Schumaker L. L., On super splines and finite elements, SIAM J. Numer. Anal. **26** (1989), 997–1005.
24. Sommer M. and Strauss H., Weak Descartes systems in generalized spline spaces, Constr. Approx. **4** (1988), 133–145.

25. Sommer M. and Strauss H., A condition of Schoenberg-Whitney type for multivariate spline interpolation, Advances in Comp. Math. **5** (1996), 381–397.
26. Sun Q., A note on the integer translates of a compactly supported distribution on $\mathbb{R}$, Arch. Math. **60** (1993), 359–363.
27. Ženíšek A., Interpolation polynomials on the triangle, Numer. Math. **15** (1970), 283–296.

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