On Almost Interpolation by Multivariate Splines

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Abstract. A survey on some recent developments in multivariate interpolation, including characterizations of almost interpolation sets with respect to finite-dimensional spaces by conditions of Schoenberg-Whitney type, is given.

1. Introduction

Let U denote a finite-dimensional subspace of real valued functions defined on some set K. The problem of describing those configurations $T = \{t_1, \ldots, t_n\} \subset K, n = \dim U$, such that for any given data $\{y_1, \ldots, y_n\}$ there exists a unique function $u \in U$ satisfying

$$u(t_i) = y_i,\ i = 1, \ldots, n,$$

has attracted considerable interest in recent years, especially for the case when $K \subset \mathbb{R}^k, k \geq 2$. In contrast to the univariate case $K \subset \mathbb{R}$, where all interpolation sets T with respect to a spline space can be characterized by the well-known Schoenberg-Whitney condition [17] (see Section 2), it seems to be no reasonably simple way to characterize interpolation sets in the multivariate case (see [6, p. 136]). Therefore, several sufficient conditions and methods to construct such configurations for multivariate interpolation have been developed (see [3, 5, 6, 15] and references therein).

A new approach to multivariate interpolation has been found by Sommer and Strauss [23] introducing the concept of almost interpolation. A set $T = \{t_1, \ldots, t_s\} \subset K, s \leq \dim U$ is called an *almost interpolation set* (*AI-set*) *with*

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respect to U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, s$ there exist points $t'_i \in B_i$ such that $T' = \{t'_1, \ldots, t'_s\}$ is an *interpolation set (I-set) with respect to U*; *i.e.*,

$$\dim U|_{T'} = s.$$

Otherwise, T' is called an *NI-set w.r.t. U*.
It is shown in [23] that for a wide class of generalized spline spaces defined on polyhedral partitions AI-sets can be characterized by conditions of Schoenberg-Whitney type (Section 3).

Davydov [8] has considered AI-sets in the case of any finite-dimensional space U of real valued functions defined on an arbitrary topological space K. Using the notion of *local dimension* (see Section 4.1) he has shown that under some minor additional hypotheses on K any U has a piecewise almost Chebyshev structure (Sections 4.2 and 4.3), and AI-sets w.r.t. U can be characterized by a Schoenberg-Whitney type condition (Section 4.4) which extends the results in [23].

In Section 5 we present some results on how to transform a given AI-set into an I-set for the case of multivariate polynomial splines.

In the sequel we shall use the notations I-set and AI-set w.r.t. a space U, respectively, as we have defined them above. We denote by $F(K)$ the linear space of all real valued functions on a topological space K and by $C(K)$ its subspace consisting of all continuous functions. Moreover, we define, for a function $u \in F(K)$

$$\operatorname{supp} u := \overline{\{t \in K : u(t) \neq 0\}},$$

and denote by card M the number of elements of a finite set M.

2. Schoenberg-Whitney Type Conditions for Univariate Spline Interpolation

In this section we shall present some well-known results on univariate spline interpolation.

Assume that $K = [a, b] \subset \mathbb{R}$ and $\Delta : a = x_0 < \ldots < x_{r+1} = b$ denote any partition on K. Let $m \in \mathbb{N}$. The *linear space of polynomial spline functions of degree m with r fixed knots* is defined by

$$U := S_m(\Delta) := \{u \in C^{m-1}[a, b] : u|_{[x_i, x_{i+1}]} \in \pi_m, 0 \leq i \leq r\}$$

where π_m denotes the linear space of polynomials of degree at most m. Then $n := \dim U = m + r + 1$ and interpolation sets w.r.t. U can be characterized by an interlacing property due to Schoenberg and Whitney [17] as follows.

Interlacing property. *If $T = \{t_1, \ldots, t_n\} \subset [a, b]$, then T is an I-set w.r.t. U if and only if*

$$t_i < x_i < t_{i+m+1}, \quad i = 1, \ldots, r. \tag{2.1}$$

An equivalent statement to (2.1) is given in terms of a basis of functions in U with minimal support, the so-called B-spline functions (see e.g. [19]).

Support property. *Let* $\{B_1, \ldots, B_n\}$ *denote the B-spline basis for* U. *If* $T = \{t_1, \ldots, t_n\} \subset [a, b]$, *then* T *is an* I*-set w.r.t.* U *if and only if*

$$t_i \in \{t \in K : B_i(t) \neq 0\}, \quad i = 1, \ldots, n. \tag{2.2}$$

A generalization of this support property to the multivariate case plays an important role in the problem of determining AI-sets, especially for locally linearly independent systems of functions (see Theorems 3.7, 4.12 and [10, Theorem 2.3]).

It is easily seen that (2.1) can be reformulated in terms of the restriction of U to certain knot intervals.

Dimension property. *If* $T = \{t_1, \ldots, t_n\} \subset [a, b]$, *then* T *is an* I*-set w.r.t.* U *if and only if*

$$\operatorname{card}(T \cap [x_i, x_j]) \leq \dim U|_{[x_i, x_j]}, \quad i, j = 0, \ldots, r, \ i < j. \tag{2.3}$$

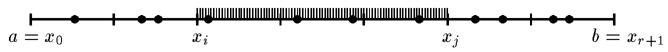

Fig. 2.1.

A property like (2.3) on the dimension behavior of U on certain "subcells" of the partition Δ will enable us to derive Schoenberg-Whitney type conditions for multivariate interpolation. In fact, for that case a more general dimension property as (2.3) will be better suitable.

Strong dimension property. *If* $T = \{t_1, \ldots, t_n\} \subset [a, b]$, *then* T *is an* I*-set w.r.t.* U *if and only if*

$$\operatorname{card}(T \cap M_P) \leq \dim U|_{M_P} \tag{2.4}$$

for any $P \subset \{0, \ldots, r\}$ *where* $M_P := \bigcup_{i \in P} [x_i, x_{i+1}]$.

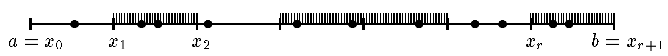

Fig. 2.2.

Remark. Schoenberg-Whitney type conditions can be used for the characterization of I-sets with respect to some other spaces of univariate functions. Interlacing property (2.1) and support property (2.2), respectively, have been extended in [16, 21] to spaces of generalized splines. An extension of the support property to locally linearly independent weak Descartes systems of functions has been found in [4]. Extensions of the dimension properties (2.3) and (2.4) to weak Chebyshev spaces have been given in [7, 9, 22].

3. Schoenberg-Whitney Type Conditions for Almost Interpolation on Polyhedral Partitions

In this section we shall present some recent results on almost interpolation of multivariate functions defined on polyhedral partitions in $\mathbb{R}^k$. The conditions which even characterize AI-sets are extensions of (2.2) and (2.4), respectively and therefore, can be considered as conditions of Schoenberg-Whitney type.

Let us begin by introducing the spaces of interest. Assume that $\mathcal{K}$ denotes a finite family of l-dimensional simplices in $\mathbb{R}^k$ where $k \in \mathbb{N}, l \in \mathbb{N} \cup \{0\}$ and $l \leq k$ satisfying the following properties:

1) If the simplex s belongs to $\mathcal{K}$, then every face of s belongs also to $\mathcal{K}$.
2) If $s, \tilde{s} \in \mathcal{K}$, then the intersection of s and $\tilde{s}$ is empty or a common face.

The point-set union of all simplices of the family $\mathcal{K}$ is called a *polyhedron* in $\mathbb{R}^k$ (see [18]).

Let

$$K := \bigcup_{i \in I} K_i$$

where every K_i is a polyhedron in $\mathbb{R}^k$ and I denotes a finite set. Assume that $K_i \not\subset \bigcup_{j \in I \setminus \{i\}} K_j$. Moreover, assume that K is *regular*; i.e., the set $\{K_i\}_{i \in I}$ of polyhedrons in K satisfies also property 2) above.

Example 3.1. *If $k = 1$, then $K = [a, b]$ and $K_i = [x_i, x_{i+1}]$, $i = 0, \ldots, r$ where $a = x_0 < \ldots < x_{r+1} = b$.*

Example 3.2. (Regular triangulation) *Let $K = \bigcup_{i \in I} K_i \subset \mathbb{R}^2$ where $\{K_i\}_{i \in I}$ is a set of triangles with the property that no vertex of K_i lies on the interior of K_j or on the interior of a side of $K_j, i, j \in I$.*

Example 3.3. (Rectangular partition) *Let $K = [a, b] \times [c, d] \subset \mathbb{R}^2$ and $a = x_0 < \ldots < x_{r+1} = b$, $c = y_0 < \ldots < y_{s+1} = d$. Then $K = \bigcup_{(i,j) \in I} K_{ij}$ where $K_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $I = \{(i, j) : i \in \{0, \ldots, r\}, j \in \{0, \ldots, s\}\}$.*

If the rectangular partition is refined by drawing in all diagonals with positive slope or both diagonals in every K_{ij}, the resulting partition is called *type-1* or *type-2 triangulation*, respectively (see [6, p. 27]).

Let $p \in \mathbb{N} \cup \{0\}$. For every $i \in I$, assume that U_i denotes a finite-dimensional subspace of $C^p(K_i)$ satisfying the *L-property*: Let $u \in U_i$ and $\tilde{t} \in K_i$ be a zero of u. If there exists $\epsilon > 0$ such that $u(t) = 0$ for every $t \in K_i$ satisfying $||\tilde{t} - t|| < \epsilon$, then $u \equiv 0$ on K_i. (In the special case when $K_i \subset \mathbb{R}$, the most important examples of U_i are Haar subspaces.)

We define the *linear space S of generalized spline functions of smoothness p* by $S := \{s \in C^p(K) :$ for every $i \in I$ there exists $s_i \in U_i$ such that $s|_{K_i} = s_i\}$. Suppose that $\{u_1, \ldots, u_n\}$ denotes a system of linearly independent functions in S. Set

$$U := \text{ span } \{u_1, \ldots, u_n\}.$$

Recently, Sommer and Strauss [23] were concerned with the question of when a subset $T = \{t_1, \ldots, t_s\}$ of $K, s \leq n$ is an AI-set w.r.t. U. For that they gave an extension of condition (2.4) as follows.

Definition 3.4. *Let $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. Then T is said to satisfy a* condition of Schoenberg-Whitney type *or T is called an* SWT-set w.r.t. U *if*

$$\operatorname{card}(T \cap \operatorname{int}_K M_P) \leq \dim U|_{M_P} \tag{3.1}$$

for any $P \subset I$ where $M_P := \bigcup_{i \in P} K_i$ and $\operatorname{int}_K M_P := K \setminus \bigcup_{i \in I \setminus P} K_i$.

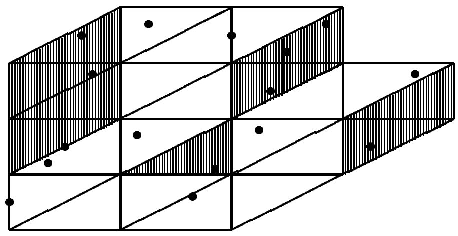

Fig. 3.1.

Using this condition a characterization of all AI–sets w.r.t. U was given in [23].

Theorem 3.5. *Let $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. Then T is an AI–set w.r.t. U if and only if T is an SWT-set w.r.t. U.*

It is a nice consequence of this result that in practice it should suffice to use AI-sets for interpolation problems. In fact, in [23] the following result was shown.

Corollary 3.6. *If $\mathcal{T} := \{T = \{t_1, \ldots, t_s\} \subset K : T$ is an AI-set w.r.t. $U\}$, and $\tilde{\mathcal{T}} := \{T \in \mathcal{T} : T$ is an NI-set w.r.t. $U\}$, then $\tilde{\mathcal{T}}$ is a set of first category in $\mathcal{T}$.*

The following result which gives an extension of (2.2) is also due to [23].

Theorem 3.7. *Let $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. The following conditions are equivalent.*

1) *T is an AI-set w.r.t. U.*
2) *For each basis $\{u_1, \ldots, u_n\}$ of U there is some permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \ldots, s.$$

4. Almost Interpolation by Functions Defined on Topological Spaces

Here we survey some results by Davydov [8], who has shown that a Schoenberg-Whitney type characterization of AI-sets holds in fact for any finite-dimensional linear space of continuous functions on a topological space satisfying some minor restrictions. Particularly, this is true for the spaces of multivariate splines with respect to non-polyhedral partitions.

4.1. Local Dimension

Assume that K is a topological space and U denotes a finite-dimensional subspace of $F(K), \dim U = n$.

Definition 4.1. *[8] Let K' be any subset of K. By the* local dimension of U on K' *we mean*

$$\text{l-dim}_{K'} U := \inf \{\dim U|_B : K' \subset B, B \text{ open } \}.$$

With the help of local dimension it is possible to give a „local" characterization of almost interpolation sets with respect to any finite-dimensional space U.

Theorem 4.2. *[8] Let $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. Then T is an AI–set w.r.t. $U \subset F(K)$ if and only if*

$$\text{l-dim}_{T'} U \geq \operatorname{card} T'$$

for any choice of a nonempty subset $T' \subset T$.

We write $\text{l-dim}_t U$ instead of $\text{l-dim}_{\{t\}} U$. The function $\varphi : K \to \mathbb{Z}_+$ defined by $\varphi(t) := \text{l-dim}_t U$ is evidently upper semicontinuous. Moreover, it is continuous on an open everywhere dense subset $G_U \subset K$. Figure 4.1 presents the graph of the local dimension of the space of univariate splines $S_m(\Delta)$.

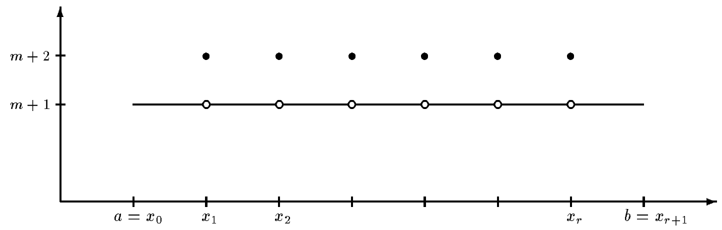

Fig. 4.1.

As another example consider the space of linear bivariate splines on the triangulation in Figure 4.2. We have

$$\text{l-dim}_{t_1} U = 3, \ \text{l-dim}_{t_2} U = 4, \ \text{l-dim}_{t_3} U = 5.$$

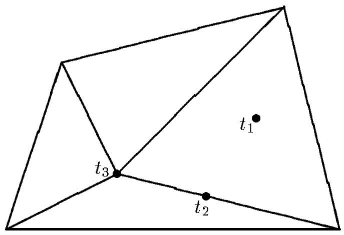

Fig. 4.2.

4.2. Almost Chebyshev Systems

It is well-known that Chebyshev systems (T-systems) play an important role in the approximation theory. Recall that a system of functions $u_1, \ldots, u_n \in F(K)$ is said to be a *Chebyshev system* if every nonzero function $u \in U =$ span $\{u_1, \ldots, u_n\}$ has at most $n-1$ zeros. The linear span of a Chebyshev system is called a *Haar space*. (Some authors prefer the notation "Chebyshev space".) It is an important feature of Haar spaces that they are as good for interpolation as possible: any set $T = \{t_1, \ldots, t_n\} \subset K$ is an interpolation set w.r.t. such a space. In fact, this property can be taken as a definition of a Haar space or a Chebyshev system.

Mairhuber's theorem [14] shows that the existence of a Haar space $U \subset F(K)$ of dimension $n \geq 2$ implies some severe restrictions on K. Particularly, K cannot be homeomorphic to a subset of $\mathbb{R}^k$, $k \geq 2$, with nonempty interior. Hence, Chebyshev systems cannot be used for approximation of multivariate functions. Because of this we consider an "almost interpolation" analogue of Chebyshev systems.

Definition 4.3. *A system of functions $u_1, \ldots, u_n \in F(K)$ is said to be an* almost Chebyshev system *if any set $T = \{t_1, \ldots, t_n\} \subset K$ is an AI–set w.r.t. $U =$ span $\{u_1, \ldots, u_n\}$. The linear span U of an almost Chebyshev system is called an* almost Haar space.

In the next theorem we give some characteristic properties of almost Haar spaces.

Theorem 4.4. *Let K be a topological space and let $U \subset F(K)$ denote a finite-dimensional linear space,* $\dim U = n$.

1) *U is an almost Haar space if and only if for any nonempty open set $B \subset K$,*

$$\dim U|_B = \min\{n, \operatorname{card} B\}.$$

2) *Suppose that every nonempty open set $B \subset K$ is infinite. Then U is an almost Haar space if and only if no nonzero function $u \in U$ can vanish identically on an open subset B of K.*
3) *Suppose that K is a compact metric space and $U \subset C(K)$. Then U is an almost Haar space if and only if it is an* almost Chebyshev subspace *of the*

normed space $C(K)$ in the sense that the set of elements $f \in C(K)$ for which there exists a unique best approximation to f from U, is of the second category in $C(K)$.

4) *Suppose that K is connected and satisfies T_1-axiom of separation. Then U is an almost Haar space if and only if* $\text{l-dim}_t\, U =$ *constant* , $t \in K$.

The notion of almost Chebyshev subspaces mentioned in 3) was introduced by Stechkin [24]. Garkavi [12, 13] showed that there exist almost Chebyshev subspaces of arbitrary finite dimensions in any separable Banach space. Parts 1) (in the case of K being a compact metric space) and 3) of the above theorem are due to Garkavi [13]. 2) is an immediate consequence of 1). 4) is proved in [8] with the help of the following result.

Proposition 4.5. *[8] Under the hypotheses of Theorem 4.4, let K' be a connected subset of K. If* $\text{l-dim}_t\, U = m,\ t \in K'$, *then* $\text{l-dim}_{K'}\, U = m$.

It is easily seen from Theorem 4.4 that the class of almost Haar spaces is rather wide. For example, any finite-dimensional space of analytic functions on a domain $K \subset \mathbb{R}^k$ is a space of this type. In the case $K \subset \mathbb{R}$ the same is true for any subspace of a Haar space.

4.3. Piecewise Almost Chebyshev Structure

Consider again the function $\varphi(t) = \text{l-dim}_t\, U$, where $U \subset F(K)$ is a finite-dimensional linear space and K denotes a topological linear space. Denote by G_U the set of all points of continuity of $\varphi(t)$ and decompose G_U into the union of its connected components,

$$G_U = \bigcup_{i \in I} K_i.$$

Then G_U is open and everywhere dense in K, so that

$$\overline{\bigcup_{i \in I} K_i} = K.$$

Because of this we consider the set $\{K_i : i \in I\}$ as a *partition* of K. The *cells* K_i of this partition are disjoint and connected.

Since $\varphi(t)$ takes only integer values, it remains constant on each cell $K_i, i \in I$. Theorem 4.4 then shows that $U|_{K_i}$ is an almost Haar space if K_i is not a singleton, satisfies T_1-axiom of separation and, additionally,

$$\text{l-dim}_t\, U = \text{l-dim}_t\, U|_{K_i},\ t \in K_i.$$

This last condition can be guaranteed by imposing some restrictions on K.

Theorem 4.6. *[8] Let K be a locally connected T_1-space and let $U \subset F(K)$ be a finite-dimensional linear space. Define the partition $K = \overline{\cup_{i\in I} K_i}$ as above. Then the following conditions hold.*

1) *The cells K_i are open and connected subsets of K.*
2) *$U|_{K_i}$ is an almost Haar space for any $i \in I$.*

Thus, under the hypotheses of Theorem 4.6, U is generated on the cells by some almost Chebyshev systems and, hence, may be thought of as a "piecewise almost Chebyshev" space.

In the case $K \subset \mathbb{R}$ we obtain a similar result without requiring that K is locally connected.

Theorem 4.7. *[8] Let K be any subset of $\mathbb{R}$ and let $U \subset F(K)$ denote a finite-dimensional linear space. Define the partition $K = \overline{\cup_{i\in I} K_i}$ as above. Then the following is true.*

1) *Each cell K_i is either a singleton or an (finite or infinite) open, closed or half-open interval. In particular, I is countable.*
2) *$U|_{K_i}$ is an almost Haar space for any $i \in I$.*

We can say more about $U|_{K_i}$ in the case when $K = [a, b]$ and $U \subset C[a, b]$ is a *weak Chebyshev space*; i.e., every nonzero function $u \in U$ has at most $n - 1$ sign changes ($n = \dim U$). By Theorem 4.6, K_i are open connected subsets of $[a, b]$, so that

$$G_U = [a, a') \cup \bigcup_{j\in J} (\alpha_j, \beta_j) \cup (b', b],$$

where $a \le a'$, $b' \le b$ (we mean $[x, x) = (x, x] = \emptyset$), (α_j, β_j), $j \in J$, are disjoint open subintervals of (a', b'), $\cup_{j\in J}(\alpha_j, \beta_j)$ is everywhere dense in (a', b').

We say that a point $t \in K$ is *essential* w.r.t. $U \subset F(K)$ if there exists $u \in U$ such that $u(t) \ne 0$.

Theorem 4.8. *Let $U \subset C[a, b]$ be a weak Chebyshev space. Suppose that any point $t \in [a, b]$ is essential w.r.t. U. Then*

$$U|_{(a,a')},\ U|_{(b',b)},\ U|_{(\alpha_j,\beta_j)},\ j \in J,$$

are Haar spaces.

Proof: Indeed, by [20, Theorem 1.4] these spaces are weak Chebyshev because U is weak Chebyshev. Theorem 4.6 states that they are also almost Haar spaces, so that no nonzero function vanishes identically on a nondegenerate proper subinterval of $[a, a']$, $[b', b]$ or $[\alpha_j, \beta_j]$ respectively. Since every point $t \in [a, b]$ is essential w.r.t. U, it follows that each of $U|_{[a,a']}$, $U|_{[b',b]}$ and $U|_{[\alpha_j,\beta_j]}$, $j \in J$, has Chebyshev rank at most $n - 1$. Then Remark i in [20, p. 59] implies that the restrictions of U to corresponding open intervals are in fact Haar spaces. ■

The statement of Theorem 4.8 can be strengthened for the important case when a weak Chebyshev subspace U of $C[a,b]$ does not contain functions with "arbitrarily small" zero intervals. Following Bartelt [1] we say that U satisfies *condition* (I) if there exists $\delta > 0$ such that if $u \in U$ and $u \equiv 0$ on $[c,d] \subset [a,b]$, $c,d \in \operatorname{supp} u \cup \{a,b\}$, then $d-c \geq \delta$. This implies a "spline-like" behavior as the following result shows.

Theorem 4.9. *[20] Let U be a weak Chebyshev subspace of $C[a,b]$ and suppose that U satisfies condition (I). The following statements hold.*

1) *There exists a finite set of points $a = x_0 < \ldots < x_{r+1} = b$ such that for each $i = 0, \ldots, r$,*
$$U|_{[x_i, x_{i+1}]}$$
is an almost Haar subspace of $C[x_i, x_{i+1}]$.

2) *If in addition every $t \in [a,b]$ is essential w.r.t. U, then there exists a finite set of points $a = x_0 < \ldots < x_{r+1} = b$ such that for each $i = 0, \ldots, r$,*
$$U|_{[x_i, x_{i+1}]}$$
is even a Haar subspace of $C[x_i, x_{i+1}]$.

4.4. Schoenberg-Whitney Type Conditions

Suppose that K is a locally connected T_1-space and $U \subset F(K)$ is a finite-dimensional linear space, $\dim U = n$. Define the partition $K = \overline{\bigcup_{i \in I} K_i}$ as in the previous subsection. Then Theorem 4.6 can be applied so that $U|_{K_i}$ is an almost Haar space when K_i is not a singleton. Assuming additionally that $U \subset C(K)$, we give a Schoenberg-Whitney type characterization of almost interpolation sets through an extension of conditions (2.4) and (3.1). The next two theorems are immediate consequences of Theorem 3.10 and Corollary 4.18 in [8].

Theorem 4.10. *Suppose that $U \subset C(K)$ and let $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. Then T is an AI-set w.r.t. U if and only if*
$$\operatorname{card}(T \cap \operatorname{int} M_P) \leq \dim U|_{M_P} \tag{4.1}$$
for any $P \subset I$ where $M_P := \overline{\cup_{i \in P} K_i}$ and $\operatorname{int} M_P$ denotes the set of all interior points of M_P w.r.t. topology on K.

When I is infinite, we have in (4.1) an infinite set of inequalities. However, we are able to show that for each fixed $T = \{t_1, \ldots, t_s\}$ it is enough to check (4.1) for a finite number of M_P's.

Theorem 4.11. *Under the hypotheses of Theorem 4.10, let $B_1, \ldots, B_s$ be open L– neighborhoods of $t_1, \ldots, t_s$, respectively; i.e.,*
$$\dim U|_{B_j} = \text{l-dim}_{t_j} U, \quad j = 1, \ldots, s.$$

In order for $T = \{t_1, \ldots, t_s\}$ *to be an AI–set w.r.t.* U *it is sufficient that (4.1) holds for any* $P \subset I$ *of the form*

$$P = \bigcup_{j \in Q} P_j, \quad Q \subset \{1, \ldots, s\},$$

where $P_j := \{i \in I : K_i \cap B_j \neq \emptyset\}, \quad j = 1, \ldots, s.$

It is easily seen that the Theorems 4.10 and 4.11 can be applied to the spaces of generalized splines considered in Section 3 as well as to any space of continuous piecewise polynomial functions with respect to an arbitrary partition of a domain $K \subset \mathbb{R}^k$.

A general version of Theorem 3.7 is also true.

Theorem 4.12. *[11] Suppose that* K *is a topological space and* $U \subset F(K)$ *is a finite-dimensional linear space,* $\dim U = n$. *Let* $T = \{t_1, \ldots, t_s\} \subset K, s \leq n$. *Then the following conditions are equivalent.*

1) T *is an AI-set w.r.t.* U.
2) *For each basis* $\{u_1, \ldots, u_n\}$ *of* U *there exists some permutation* σ *of* $\{1, \ldots, n\}$ *such that* $t_i \in \operatorname{supp} u_{\sigma(i)}$, *for all* $i = 1, \ldots, s$.

5. Transforming AI-sets into I-set

In the preceding sections we have considered the problem of characterizing AI-set w.r.t. finite-dimensional subspaces U of $F(K)$.

By Corollary 3.6 we know, at least for spaces of generalized splines U, that if $\mathcal{T}$ denotes the set of all AI-sets w.r.t. U and $\tilde{\mathcal{T}}$ its subset of NI-sets w.r.t. U, then $\tilde{\mathcal{T}}$ is a set of first category in $\mathcal{T}$.

Hence the question arises whether it is possible to find simple methods for transforming AI-sets T into I-sets in some neighborhood of T.

Let $T = \{t_1, \ldots, t_s\}$, $s \leq n$, $n = \dim U$, be an AI-set w.r.t. U and let some neighborhoods $B_1, \ldots, B_s$ of the points $t_1, \ldots, t_s$, respectively, be given. Set $n_i := \dim U|_{B_i}$, $i = 1, \ldots, s$. It is always possible to choose some points $t_{i,j} \in B_i$, $j = 1, \ldots, n_i, i = 1, \ldots, s$, in such a way that $T_i := \{t_{i,1}, \ldots, t_{i,n_i}\}$ is an I-set w.r.t. $U|_{B_i}$, $i = 1, \ldots, s$.

Theorem 5.1. *[8] For any* $i \in \{1, \ldots, s\}$ *there exists* $\mu(i) \in \{1, \ldots, n_i\}$ *such that* $\{t_{i,\mu(i)} : i = 1, \ldots, s\}$ *is an I-set w.r.t.* U.

Assume now that S denotes the linear space of polynomial splines of smoothness p and degree m defined as in Section 3 on the set $K = \cup_{i \in I} K_i \subset \mathbb{R}^k$ where K_i is a convex polyhedron for all $i \in I$, such that $S|_{K_i}$ coincides with the set of polynomials of total degree at most m, $i \in I$. We consider the following situation.

Let a set $T = \{t_1, \ldots, t_n\} \subset \mathbb{R}^k$ where $n = \dim S$ be given. Assume that T is an AI-set w.r.t. S satisfying $t_i \in K_{j_i}$, $i = 1, \ldots, n$. Moreover, let $V = \{v_1, \ldots, v_n\}$

be an I-set w.r.t. S such that $v_i \in K_{j_i}$, $i = 1, \ldots, n$. Notice that by the definition of AI-sets every neighborhood of T contains such an I-set. We now define the straight lines through t_i and v_i,

$$l_i := \{t \in K_{j_i} : \text{ there exists } \lambda \in \mathbb{R} \text{ such that } t = t_i(\lambda) = (1-\lambda)t_i + \lambda v_i\}.$$

Since K_{j_i} is convex, we have $t_i(\lambda) \in K_{j_i}$ for all $0 \le \lambda \le 1$.

Under these assumptions we obtain the following result.

Theorem 5.2. *[22] Let $T(\lambda) := \{t_1(\lambda), \ldots, t_n(\lambda)\}$. Then $T(\lambda)$ is an I-set w.r.t. S for all $0 \le \lambda \le 1$ with the exception of a finite number of points $0 \le \lambda_1 < \ldots \lambda_q \le 1$ where $0 \le q \le mn$.*

Corollary 5.3. *[22] Let the assumptions of Theorem 5.2 be given. Then there exists a real number $\lambda_0 > 0$ such that $T(\lambda)$ are I-sets w.r.t. S for all $0 < \lambda \le \lambda_0$.*

Remark 5.4. 1) Let $V = \{v_1, \ldots, v_n\}$ be an I-set w.r.t. S such that $v_i \in K_{j_i}$. Then it follows from Theorem 3.5 that every set $T = \{t_1, \ldots, t_n\}$ satisfying $t_i \in K_{j_i}$, $i = 1, \ldots, n$, is an AI-set w.r.t. S. Hence we choose an arbitrary set $\tilde{T} = \{\tilde{t_1}, \ldots, \tilde{t_n}\}$ satisfying $\tilde{t_i} \in K_{j_i}$, $i = 1, \ldots, n$. It follows from Corollary 5.3 that there is a real number $\lambda_0 > 0$ such that $\{(1-\lambda)\tilde{t}_i + \lambda v_i\}_{i=1}^n$ is an I–set w.r.t. S for all $0 < \lambda < \lambda_0$. This means the following: If we have an I–set V such that $v_i \in K_{j_i}$, $i = 1, \ldots, n$, then we can move the points v_i to arbitrary points $\tilde{t}_i$ in the same polyhedron and we always have I–sets on the lines connecting v_i and $\tilde{t}_i$ in a neighborhood of $\tilde{t}_i$, $i = 1, \ldots, n$. But this is not true if both T and V are AI–sets which fail to be I–sets. It can be shown by simple examples that $\{(1-\lambda)t_i + \lambda v_i\}_{i=1}^n$ can be NI–sets for all $0 \le \lambda \le 1$. Therefore, starting with some special interpolation configuration (a variety of methods of constructing them can be found in [3, 5, 6, 15]), we can apply the above method in order to obtain interpolation configurations with desirable location. For example, for the space of continuous bivariate spline functions on regular triangulations an initial I–set can be easily constructed by well-known finite-element methods (see e.g. [2, p. 155]).

2) Let us consider the case of Theorem 5.2 such that V is an I-set and T is an AI-set. We shall give an example where $T(\lambda)$ is an NI-set for some $0 \le \lambda < 1$: Define a set of vertices in $\mathbb{R}^2$ by $e_1 = (1,0), e_2 = (0,1), e_3 = (-1,0)$ and $e_4 = (0,-1)$ and let $K = K_1 \cup K_2$ be a triangulation such that K_1 is the convex hull of $\{e_1, e_2, e_4\}$ and K_2 is the convex hull of $\{e_2, e_3, e_4\}$. Assume that S is the space of linear continuous splines defined on K. Then the set $V = \{e_1, \ldots, e_4\}$ is an I-set w.r.t. S. The set $T = \{t_1, \ldots, t_4\}$ given by $t_1 = (1/2, 0), t_2 = (1/2, -1/2), t_3 = (-1/2, 0)$ and $t_4 = (-1/2, 1/2)$ is an AI-set, but fails to be an I-set. We now define the lines

$$t_i(\lambda) = (1-\lambda)t_i + \lambda e_i,\ i = 1, \ldots, 4.$$

For $\lambda = 1/3$ all points $t_i(1/3)$, $i = 1, \ldots, 4$ are contained in the x-axis. Hence $\{t_i(\lambda)\}_{i=1}^4$ is an NI-set for $\lambda = 0$ and $\lambda = 1/3$. It is easily seen that for any other $\lambda \in (0, 1/3) \cup (1/3, 1]$, $\{t_i(\lambda)\}_{i=1}^4$ is an I-set.

Finally we give a description of a wide class of I-sets for linear bivariate splines.

Theorem 5.5. *[22] Let $K = \cup_{i\in I} K_i \subset \mathbb{R}^2$ be a regular triangulation and S denote the space of linear continuous splines on K. Assume that $\{e_1, \ldots, e_n\}$ denotes the set of vertices of K. Then $S = \operatorname{span}\{u_1, \ldots, u_n\}$ where $u_i \in S$ is defined by $u_i(e_j) = \delta_{ij}$, $i, j = 1, \ldots, n$. Let us define a set M_i by*

$$M_i := \{t \in K : u_i(t) > \frac{1}{2}\},\ i = 1, \ldots, n.$$

Then every set $\{t_1, \ldots, t_n\}$ satisfying $t_i \in M_i$ for $i = 1, \ldots, n$ is an I-set w.r.t. S.

Note that the result is no longer true if we replace each set M_i by its closure.

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