Locally Stable Spline Bases on Nested Triangulations

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Abstract. Given a nested sequence of triangulations $\triangle_0, \triangle_1, \ldots$ of a polygonal domain Ω, we construct for any $r \geq 1$, $d \geq 4r+1$, locally stable bases for some spaces $\tilde{\mathcal{S}}_d^r(\triangle_0) \subset \tilde{\mathcal{S}}_d^r(\triangle_1) \subset \cdots \subset \tilde{\mathcal{S}}_d^r(\triangle_n) \subset \cdots$ of bivariate polynomial splines of smoothness r and degree d. In particular, the bases are stable and locally linearly independent simultaneously.

§1. Introduction

Let

$$\mathcal{S}_d^r(\triangle) = \{s \in C^r(\Omega) : \; s|_T \in \mathcal{P}_d \;\text{ for all triangles } T \in \triangle\},$$

be the space of polynomial splines of degree d and smoothness r on a triangulation $\triangle$ of a polygonal domain Ω, where $\mathcal{P}_d$ is the space of bivariate polynomials of total degree d.

Suppose the sequence of triangulations $\triangle_0, \triangle_1, \ldots, \triangle_n, \ldots$, of Ω obtained by consecutive refinements of an initial coarse triangulation $\triangle_0$ is shape regular, *i.e.*, the smallest angle of all triangles in each $\triangle_n$ is at least θ, for some $\theta > 0$ independent of n. Standard conforming smooth finite elements [4] with respect to $\triangle_n$ span the superspline subspaces

$$\mathcal{S}_d^{r,2r}(\triangle) = \{s \in \mathcal{S}_d^r(\triangle) : \; s \in C^{2r}(v) \;\text{ for all vertices } v \text{ of } \triangle\},$$

of $\mathcal{S}_d^r(\triangle)$, $r \geq 4r+1$, [3,21], where $s \in C^{2r}(v)$ means that s is $2r$ times differentiable at v. Since these spaces are not nested, *i.e.*,

$$\mathcal{S}_d^{r,2r}(\triangle_n) \not\subset \mathcal{S}_d^{r,2r}(\triangle_{n+1}),$$

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there are well known complications in using them for multiscale numerical methods, see *e.g.* [19]. As an alternative, it was suggested in [5,17] to use locally supported bases for the full spline spaces $\mathcal{S}_d^r(\triangle_n)$. However, the standard constructions [1,14,15,18] of locally supported bases for $\mathcal{S}_d^r(\triangle)$ in general lack the highly desirable property of stability, where a sequence of spline bases $(s_i^{[n]})_{i\in I_n} \subset \mathcal{S}_d^r(\triangle_n)$, $n = 0, 1, \ldots$, is called (L_p-)stable if for all choices of the coefficient vectors $c = (c_i)_{i\in I_n}$,

$$K_1\|c\|_p \;\le\; \|\sum_{i\in I_n} c_i s_i^{[n]}\|_{L_p(\Omega)} \;\le\; K_2\|c\|_p, \tag{1}$$

with constants K_1, K_2 depending only on r, d and θ. On the other hand, stable locally supported spline bases used in [2,16] span superspline subspaces of $\mathcal{S}_d^r(\triangle)$, $d \ge 3r+2$, that are less restrictive than $\mathcal{S}_d^{r,2r}(\triangle)$ but also lack nestedness for nested triangulations.

Recently constructed [10,11] stable locally supported bases for $\mathcal{S}_d^r(\triangle)$, $d \ge 3r+2$, solve this problem. Moreover, a construction of stable locally supported bases for the full spline spaces over regular triangulations of polyhedral domains in $n \ge 3$ variables is also available [7] if $d \ge r2^n + 1$. A certain drawback of these constructions is the fact that the dimension of $\mathcal{S}_d^r(\triangle)$ is *instable* since it depends on the geometry of the triangulation [18,20], thus allowing sudden changes in the number of basis functions as certain vertices are slightly perturbed. Computational aspects of this situation were discussed in [8] and numerical evidence was provided showing that despite the instability of the dimension the basis splines can be efficiently computed. However, it may be desirable to work with *nested subspaces of $\mathcal{S}_d^r(\triangle_n)$ with stable dimension.*

In this paper we present a construction of locally supported bases $(s_i^{[n]})_{i\in I_n}$ for certain nested subspaces $\tilde{\mathcal{S}}_d^r(\triangle_n)$ of $\mathcal{S}_d^r(\triangle_n)$ if $d \ge 4r+1$. These subspaces satisfy

$$\mathcal{S}_d^{r,2r}(\triangle_n) \subset \tilde{\mathcal{S}}_d^r(\triangle_n) \subset \mathcal{S}_d^r(\triangle_n), \qquad n = 0, 1, \ldots, \tag{2}$$

and therefore include all polynomials in $\mathcal{P}_d$. The dimension of $\tilde{\mathcal{S}}_d^r(\triangle_n)$ is independent of the geometry of $\triangle_n$. Moreover, the sequence of bases is locally L_p-stable in the sense that for all choices of the coefficient vectors $c = (c_i)_{i\in I_n}$, and for each triangle $T \in \triangle_n$,

$$K_1\|c|_{I_n(T)}\|_p \;\le\; \|\sum_{i\in I_n(T)} c_i s_i^{[n]}\|_{L_p(T)} \;\le\; K_2\|c|_{I_n(T)}\|_p, \tag{3}$$

with constants K_1, K_2 depending only on r, d and θ, where

$$I_n(T) := \{i \in I_n :\; T \subset \operatorname{supp} s_i^{[n]}\}.$$

Obviously, a locally L_p-stable basis is always locally linearly independent, *i.e.*, for each $T \in \triangle_n$, the functions in

$$\{s_i^{[n]}|_T : \ i \in I_n(T)\}$$

are linearly independent. Since $s_i^{[n]}|_T$ coincide with some polynomials in $\mathcal{P}_d$, it follows that $\#I_n(T) \le \binom{d+2}{2}$, and hence, a locally L_p-stable spline basis is also L_p*-stable* in the sense of (1).

We recall that local stability is a property of B-splines and finite-element bases, whereas it was shown in [10,11] that stability and local linear independence are not compatible in general for the bases of the full spline space $\mathcal{S}_d^r(\triangle)$ in two variables.

As a simple example to show that local linear independence and stability together do not necessarily imply local stability, consider the univariate spaces $S_n = \operatorname{span}\{s_1^{[n]}, s_2^{[n]}, s_3^{[n]}\}$, $n = 1, 2, \ldots$, where

$$s_1^{[n]} = \begin{cases} t, & t \in [0,1], \\ 0, & t \in [-1,0], \end{cases} \qquad s_2^{[n]} = \begin{cases} 0, & t \in [0,1], \\ -t, & t \in [-1,0], \end{cases}$$

$$s_3^{[n]} = \begin{cases} (1-t)/n, & t \in [0,1], \\ -(1+t)(t-1/n), & t \in [-1,0]. \end{cases}$$

(For more on locally linearly independent bases, see *e.g.* [6,12,13].)

§2. Spaces $\tilde{\mathcal{S}}_d^r(\triangle_n)$ and Construction of Bases

Let $\mathcal{V}_n$ and $\mathcal{E}_n$ be the sets of all vertices and all edges of the triangulation $\triangle_n$, respectively. Denote by E_n the union of all edges $e \in \mathcal{E}_n$. Since the triangulations $\triangle_n$, $n = 1, 2, \ldots$, are obtained by consecutive refinements of $\triangle_0$, we have

$$\mathcal{V}_0 \subset \mathcal{V}_1 \subset \cdots \subset \mathcal{V}_n \subset \cdots,$$
$$E_0 \subset E_1 \subset \cdots \subset E_n \subset \cdots.$$

Let

$$\tilde{\mathcal{V}}_0 = \emptyset, \qquad \tilde{\mathcal{V}}_n := \tilde{\mathcal{V}}_{n-1} \cup \big[(\mathcal{V}_n \setminus \mathcal{V}_{n-1}) \cap E_{n-1} \cap \operatorname{int}\Omega\big], \quad n = 1, 2, \ldots.$$

For any $v \in \bigcup_{n \in \mathbb{N}} \tilde{\mathcal{V}}_n$, let $n_v := \min\{n : \ v \in \tilde{\mathcal{V}}_n\}$. Obviously, there is a unique edge e_v of $\triangle_{n_v - 1}$, with adjacent triangles $T_v^+, T_v^- \in \triangle_{n_v-1}$, such that v lies in the interior of e_v. We set

$$\tilde{\mathcal{S}}_d^r(\triangle_n) = \{s \in \mathcal{S}_d^r(\triangle_n) : \ s \in C^{2r}(v) \ \text{ for all } v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n, \text{ and}$$
$$s|_{T_v^+} \in C^{2r}(v), \ s|_{T_v^-} \in C^{2r}(v) \ \text{ for all } v \in \tilde{\mathcal{V}}_n\}.$$

It is easy to see that (2) is satisfied and that the spaces are nested,

$$\tilde{\mathcal{S}}_d^r(\triangle_{n-1}) \subset \tilde{\mathcal{S}}_d^r(\triangle_n), \qquad n = 1, 2, \ldots. \tag{4}$$

To show the latter, we suppose that $s \in \tilde{\mathcal{S}}_d^r(\triangle_{n-1})$. Then $s \in \mathcal{S}_d^r(\triangle_{n-1}) \subset \mathcal{S}_d^r(\triangle_n)$. If $v \in \tilde{\mathcal{V}}_n$, then either $v \in \tilde{\mathcal{V}}_{n-1}$, or $n_v = n$, $T_v^+, T_v^- \in \triangle_{n-1}$ and v lies in the interior of the common edge e_v of these two triangles. Obviously, in both cases $s|_{T_v^+} \in C^{2r}(v)$, $s|_{T_v^-} \in C^{2r}(v)$. If, otherwise, $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, then either $v \in \mathcal{V}_{n-1} \setminus \tilde{\mathcal{V}}_{n-1}$, or $v \in \Omega \setminus E_{n-1}$, or $v \in \partial\Omega \setminus \mathcal{V}_{n-1}$. In all three cases $s \in C^{2r}(v)$.

We first determine a basis for the dual space $(\tilde{\mathcal{S}}_d^r(\triangle_n))^*$ by using usual nodal functionals. Let D_e and $D_{e^\perp}$ denote the derivative in the direction parallel or perpendicular to an edge $e \in \mathcal{E}_n$, respectively. We will also need the same notation $D_{e_v}, D_{e_v^\perp}$ for the special edge e_v defined above for each $v \in \tilde{\mathcal{V}}_n$. The linear functional evaluating at $\xi \in \Omega$ any function f continuous at ξ will be denoted by δ_ξ.

Consider the set

$$\mathcal{N} = \Big(\bigcup_{v \in \mathcal{V}_n} \mathcal{N}_v\Big) \cup \Big(\bigcup_{e \in \mathcal{E}_n} \mathcal{N}_e\Big) \cup \Big(\bigcup_{T \in \triangle_n} \mathcal{N}_T\Big),$$

of nodal linear functionals on $\tilde{\mathcal{S}}_d^r(\triangle_n)$, where for each $T = \langle v_1, v_2, v_3 \rangle \in \triangle_n$,

$$\mathcal{N}_T = \{\delta_\xi : \ \xi \in \Xi_T\},$$
$$\Xi_T = \{\xi = \frac{i_1 v_1 + i_2 v_2 + i_3 v_3}{d} : \ i_1 + i_2 + i_3 = d, \quad i_1, i_2, i_3 > r\},$$

for each edge $e = \langle v_1, v_2 \rangle \in \mathcal{E}_n$,

$$\mathcal{N}_e = \{\delta_\xi D_{e^\perp}^q : \ \xi \in \Xi_{e,q}, \quad q = 0, \ldots, r\},$$
$$\Xi_{e,q} = \{\xi = \frac{i_1 v_1 + i_2 v_2}{d} : \ i_1 + i_2 = d, \quad i_1, i_2 > 2r - q\},$$

and for each vertex $v \in \mathcal{V}_n$ the set $\mathcal{N}_v$ is defined as follows. If $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, then

$$\mathcal{N}_v = \{\delta_v D_x^\alpha D_y^\beta : \ \alpha + \beta \le 2r\}.$$

If $v \in \tilde{\mathcal{V}}_n$, then

$$\begin{aligned}\mathcal{N}_v = &\{\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta : \ \alpha + \beta \le 2r, \ \beta \le r\} \\ &\cup \{\delta_v^+ D_{e_v}^\alpha D_{e_v^\perp}^\beta : \ \alpha + \beta \le 2r, \ \beta \ge r + 1\} \\ &\cup \{\delta_v^- D_{e_v}^\alpha D_{e_v^\perp}^\beta : \ \alpha + \beta \le 2r, \ \beta \ge r + 1\},\end{aligned}$$

where $\delta_v^\pm f := \delta_v(f|_{T_v^\pm})$.

Theorem 1. *The set $\mathcal{N}$ is a basis for $(\tilde{\mathcal{S}}_d^r(\triangle_n))^*$.*

Proof: We first prove that $\mathcal{N}$ is a spanning set for $(\tilde{\mathcal{S}}_d^r(\triangle_n))^*$. This will follow if we show that $\eta s = 0$ for all $\eta \in \mathcal{N}$ implies $s = 0$ whenever $s \in \tilde{\mathcal{S}}_d^r(\triangle_n)$. Given a triangle $T \in \triangle_n$, with vertices v_1, v_2, v_3 and edges e_1, e_2, e_3, consider the set

$$\mathcal{N}(T) = \Big(\bigcup_{i=1}^{3} \mathcal{N}_{v_i}(T)\Big) \cup \Big(\bigcup_{i=1}^{3} \mathcal{N}_{e_i}\Big) \cup \mathcal{N}_T, \tag{5}$$

with $\mathcal{N}_v(T) := \mathcal{N}_v$ if $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, and

$$\begin{aligned}\mathcal{N}_v(T) := &\{\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta \,:\, \alpha+\beta \le 2r,\ \beta \le r\} \\ &\cup \begin{cases} \{\delta_v^+ D_{e_v}^\alpha D_{e_v^\perp}^\beta \,:\, \alpha+\beta \le 2r,\ \beta \ge r+1\}, & \text{if } T \subset T_v^+, \\ \{\delta_v^- D_{e_v}^\alpha D_{e_v^\perp}^\beta \,:\, \alpha+\beta \le 2r,\ \beta \ge r+1\}, & \text{if } T \subset T_v^-, \end{cases}\end{aligned}$$

if $v \in \tilde{\mathcal{V}}_n$. Clearly, $\eta s = 0$ for all $\eta \in \mathcal{N}(T)$ implies that the polynomial $p_T = s|_T$ satisfies homogeneous interpolation conditions of the standard interpolation scheme used to define finite elements of the space $\mathcal{S}_d^{r,2r}(\triangle)$, see *e.g.* [21]. Therefore $s|_T = 0$ for each T, and $s = 0$.

We now prove that $\mathcal{N}$ is linearly independent. To this end, we show that for any $a = (a_\eta)_{\eta\in\mathcal{N}} \in \mathbb{R}^{\#\mathcal{N}}$, there exists a spline $s \in \tilde{\mathcal{S}}_d^r(\triangle_n)$ such that

$$\eta s = a_\eta, \qquad \text{all } \eta \in \mathcal{N}. \tag{6}$$

We define the polynomial pieces p_T, $T \in \triangle_n$, using the finite-element interpolation scheme mentioned above, such that for each $T \in \triangle_n$,

$$\eta p_T = a_\eta, \qquad \text{all } \eta \in \mathcal{N}(T). \tag{7}$$

Setting $s|_T = p_T$ for all $T \in \triangle_n$, we have to show that the piecewise polynomial function s lies in $\tilde{\mathcal{S}}_d^r(\triangle_n)$.

Let $v \in \mathcal{V}_n \setminus \tilde{\mathcal{V}}_n$, and let T_1, T_2 are any two triangles attached to v. Since $\mathcal{N}_v \subset \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$, we have by (7),

$$\delta_v D_x^\alpha D_y^\beta p_{T_1} = \delta_v D_x^\alpha D_y^\beta p_{T_2}, \qquad \text{all } \alpha+\beta \le 2r,$$

which ensures that $s \in C^{2r}(v)$. If $v \in \tilde{\mathcal{V}}_n$, then for any two triangles T_1, T_2 attached to v,

$$\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta p_{T_1} = \delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta p_{T_2}, \qquad \text{all } \alpha+\beta \le 2r,\ \beta \le r.$$

Moreover, if either $T_1 \cup T_2 \subset T_v^+$ or $T_1 \cup T_2 \subset T_v^-$, then

$$\delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta p_{T_1} = \delta_v D_{e_v}^\alpha D_{e_v^\perp}^\beta p_{T_2}, \qquad \text{all } \alpha+\beta \le 2r,$$

which implies that $s|_{T_v^+} \in C^{2r}(v)$ and $s|_{T_v^-} \in C^{2r}(v)$.

Let $e = \langle v_1, v_2 \rangle$ be an interior edge of $\triangle_n$, with adjacent triangles T_1, T_2. From the above it follows that for all $\alpha + \beta \le 2r$, $\beta \le r$,

$$\delta_{v_1} D_e^\alpha D_{e^\perp}^\beta p_{T_1} = \delta_{v_1} D_e^\alpha D_{e^\perp}^\beta p_{T_2}, \qquad \delta_{v_2} D_e^\alpha D_{e^\perp}^\beta p_{T_1} = \delta_{v_2} D_e^\alpha D_{e^\perp}^\beta p_{T_2}.$$

Moreover, since $\mathcal{N}_e \subset \mathcal{N}(T_1) \cap \mathcal{N}(T_2)$, we have

$$\delta_\xi D_{e^\perp}^q p_{T_1} = \delta_\xi D_{e^\perp}^q p_{T_2}, \qquad \xi \in \Xi_{e,q}, \quad q = 0, \ldots, r.$$

This implies that for each $q = 0, \ldots, r$, the univariate polynomial

$$\tilde{p} := D_{e^\perp}^q (p_{T_1} - p_{T_2})|_e$$

of degree $d - q$ satisfies homogeneous Hermite interpolation conditions

$$\begin{aligned} &D_e^\mu \tilde{p}(v_1) = D_e^\mu \tilde{p}(v_2) = 0, \qquad \mu = 0, \ldots, 2r - q, \\ &\tilde{p}(\xi) = 0, \qquad \text{all } \xi \in \Xi_{e,q}. \end{aligned}$$

Therefore, $\tilde{p} = 0$, *i.e.*,

$$D_{e^\perp}^q p_{T_1}|_e = D_{e^\perp}^q p_{T_2}|_e, \qquad q = 0, \ldots, r,$$

which shows that s is r times continuously differentiable across e. Since $s \in \tilde{\mathcal{S}}_d^r(\triangle_n)$, (6) follows from (7), and the proof is complete. □

By counting the number of functionals in each $\mathcal{N}_v$, $\mathcal{N}_e$, and $\mathcal{N}_T$, it is easy to check the following dimension formula,

$$\begin{aligned} \dim \tilde{\mathcal{S}}_d^r(\triangle_n) = &\binom{d - 3r - 1}{2} \# \triangle_n + \frac{(r+1)(2d - 7r - 2)}{2} \# \mathcal{E}_n \\ &+ \binom{2r+2}{2} \# \mathcal{V}_n + \binom{r+1}{2} \# \tilde{\mathcal{V}}_n, \end{aligned} \tag{8}$$

which shows that the dimension is independent of geometry.

The desired basis $(s_\eta^{[n]})_{\eta \in \mathcal{N}}$ for $\tilde{\mathcal{S}}_d^r(\triangle_n)$ can be determined by the duality condition

$$\mu s_\eta^{[n]} = \delta_{\mu,\eta}, \qquad \text{all } \mu, \eta \in \mathcal{N}. \tag{9}$$

Arguing similarly to the first part of the proof of Theorem 1, it is easy to show that

$$s_\eta^{[n]}|_T \ne 0 \quad \text{only if} \quad \eta \in \mathcal{N}(T), \tag{10}$$

which in particular implies that

$$\operatorname{supp} s_\eta^{[n]} \subset \begin{cases} T, & \text{if } \eta \in \mathcal{N}_T \text{ for some } T \in \mathcal{T}_n, \\ \operatorname{star}(e), & \text{if } \eta \in \mathcal{N}_e \text{ for some } e \in \mathcal{E}_n, \\ \operatorname{star}(v), & \text{if } \eta \in \mathcal{N}_v \text{ for some } v \in \mathcal{V}_n, \end{cases} \tag{11}$$

where star(e) and star(v), respectively, denote the union of all triangles in $\triangle_n$ attached to an edge e or vertex v, respectively.

Theorem 2. *The above constructed sequence of bases* $(s_\eta^{[n]})_{\eta\in\mathcal{N}}$, $n = 0,1,\ldots$, *is locally* L_p*-stable,* $1 \le p \le \infty$, *after a suitable renorming.*

Proof: By (10), we have $I_n(T) = \mathcal{N}(T)$, where

$$I_n(T) := \{\eta \in \mathcal{N} :\ T \subset \operatorname{supp} s_\eta^{[n]}\}.$$

Since $\#\mathcal{N}(T) = \binom{d+2}{2}$, we have

$$\|\sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_\infty(T)} \le \binom{d+2}{2} \max_{\eta\in\mathcal{N}(T)} \|c_\eta s_\eta^{[n]}\|_{L_\infty(T)}.$$

The functionals in $\mathcal{N}(T)$ constitute a standard finite-element interpolation scheme, see *e.g.* [21]. Therefore, the general estimates for the norms of the finite-element interpolation operators [4] imply that

$$\|s_\eta^{[n]}\|_{L_\infty(T)} \le K_1 h_T^{q(\eta)}, \tag{12}$$

where K_1 depends only on r, d and θ, h_T denotes the diameter of T, and $q(\eta)$ is the order of the derivative that defines η. On the other hand, by the Markov inequality, we have for each $\mu \in \mathcal{N}(T)$,

$$|c_\mu| = \mu\Big(\sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\Big) \le K_2 h_T^{-q(\eta)} \|\sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_\infty(T)}, \tag{13}$$

where K_2 depends only on d and θ. Therefore,

$$K_2^{-1} \max_{\eta\in\mathcal{N}(T)} |c_\eta| \le \|\sum_{\eta\in\mathcal{N}(T)} c_\eta h_T^{-q(\eta)} s_\eta^{[n]}\|_{L_\infty(T)} \le K_1 \binom{d+2}{2} \max_{\eta\in\mathcal{N}(T)} |c_\eta|.$$

Since $T \subset \operatorname{supp} s_\eta^{[n]}$ and $\operatorname{supp} s_\eta^{[n]}$ is at most the star of a vertex, we have

$$h_T \le h_\eta \le K_3 h_T, \tag{14}$$

where h_η denotes the diameter of $\operatorname{supp} s_\eta^{[n]}$, and K_3 depends only on θ (see [16]). Therefore, we conclude that the sequence of bases

$$(h_\eta^{-q(\eta)} s_\eta^{[n]})_{\eta\in\mathcal{N}}, \qquad n = 0,1,\ldots,$$

is locally L_∞-stable, which completes the proof for the case $p = \infty$.

Let $1 \le p < \infty$. In this case we set

$$\hat{s}_\eta^{[n]} := A_\eta^{-1/p} h_\eta^{-q(\eta)} s_\eta^{[n]},$$

where A_η is the area of $\operatorname{supp} s_\eta^{[n]}$. Then

$$A_T \le A_\eta \le K_4 A_T, \tag{15}$$

where A_T is the area of T, and K_4 depends only on θ. We have

$$\begin{aligned}\| \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_p(T)}^p &= \int_T | \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}|^p \\ &\le A_T \|s_\eta^{[n]}\|_{L_\infty(T)}^p (\#\mathcal{N}(T))^{p-1} \sum_{\eta\in\mathcal{N}(T)} |c_\eta|^p,\end{aligned}$$

which by (12) implies that

$$\| \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_p(T)} \le K_5 A_T^{1/p} h_T^{q(\eta)} \|c|_{\mathcal{N}(T)}\|_p, \tag{16}$$

with a constant K_5 depending only on r, d and θ. On the other hand, since

$$\sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}|_T$$

is a polynomial of degree d, we have for any $\mu \in \mathcal{N}(T)$ by (13) and a Nikolskii-type inequality,

$$\begin{aligned}|c_\mu| &\le K_2 h_T^{-q(\eta)} \| \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_\infty(T)} \\ &\le K_6 A_T^{-1/p} h_T^{-q(\eta)} \| \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_p(T)},\end{aligned}$$

where K_6 depends only on d and θ. Therefore,

$$\begin{aligned}\|c|_{\mathcal{N}(T)}\|_p &= (\sum_{\mu\in\mathcal{N}(T)} |c_\mu|^p)^{1/p} \\ &\le (\#\mathcal{N}(T))^{1/p} K_6 A_T^{-1/p} h_T^{-q(\eta)} \| \sum_{\eta\in\mathcal{N}(T)} c_\eta s_\eta^{[n]}\|_{L_p(T)}.\end{aligned}$$

Since $\#\mathcal{N}(T) = \binom{d+2}{2}$, this last inequality together with (14)–(16) imply the local L_p-stability of the sequence of bases

$$(\hat{s}_\eta^{[n]})_{\eta\in\mathcal{N}}, \qquad n = 0, 1, \ldots. \quad \square$$

Finally, we note that the stability of the bases and the fact that

$$\mathcal{P}_d \subset \tilde{\mathcal{S}}_d^r(\triangle_n), \qquad n = 0, 1, \ldots,$$

can be used in a standard way (see[16]) to show the optimal approximation power of the spaces $\tilde{\mathcal{S}}_d^r(\triangle_n)$.

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