On Local Refinement of Smooth Finite Elements and Splines

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Abstract

We present an algorithm of local refinement of finite-element and spline approximations based on a uniform refinement of the mesh. Since a local mesh refinement is not needed, the method is in particular applicable to low order smooth polynomial splines on uniform type triangulations, tensor product and box splines as well as some smooth composite finite elements. The algorithm can be used in the context of adaptive approximation methods.

1 Introduction

A typical adaptive approximation algorithm requires local refinement of the approximation spaces, i.e. the initial coarse space has to be succesively enhanced in the regions where its approximation power is insufficient, thus producing a sequence of spaces $S^{[0]}, S^{[1]}, \ldots S^{[n]}$, where the last one, $S^{[n]}$, is capable to reproduce the unknown function (say, the solution of a partial differential equation or the function whose values are only known at certain points scattered over a bounded domain $\Omega \subset \mathbb{R}^d$) with required accuracy by an approximation method. Moreover, in order to take advantage of multilevel techniques, it is desirable to maintain the *nested* structure of the spaces, i.e. the refined space should contain all elements of the original space, such that

$$S^{[0]} \subset S^{[1]} \subset \cdots \subset S^{[n]}.$$

Local refinement is an easy task if $S^{[m]}$ are the spaces of piecewise polynomials on a partition of Ω without any specific requirements on how the neighboring polynomial pieces are attached to each other, in particular without any continuity or smoothness conditions. Indeed, it is sufficient in this case to locally refine the mesh by any method. For *continuous* piecewise polynomials on triangulations there are algorithms of local mesh refinement based on bisection.

There are specific difficulties with local mesh refinement for the spaces of *smooth* piecewise polynomials needed for the numerical treatment of higher order partial differential equations and for multiresolution methods of function approximation and modeling. The meshs needed for the classical smooth composite elements (Clough-Tocher element, Powell-Sabin element and their generalizations) are not compatible with bisection, whereas the multilevel spaces generated by polynomial finite elements (e.g. Argyris element), are not nested. This problem can be overcome by considering the space of all piecewise polynomials of given degree q and smoothness r (the *spline space* $S_q^r(\triangle)$) with respect to a triangulation $\triangle$ instead of its subspace generated by finite elements. The spaces $S_q^r(\triangle_0), S_q^r(\triangle_1), \ldots, S_q^r(\triangle_m), \ldots$ are nested if the triangulations $\triangle_0, \triangle_1, \ldots, \triangle_m, \ldots$ are obtained by consecutive refinements. If $q \geq r2^d + 1$ (even $q \geq 3r + 2$ for $d = 2$), these spaces have *stable local bases* [1, 4, 5], which is a prerequisite of efficient multilevel techniques. Although the dimension of the spaces $S_q^r(\triangle)$ is not stable as it can suddently change when some exceptional triangulations are slightly deformed, the stable bases can still be efficiently computed [2]. Alternatively, if $d = 2$ and $q \geq 4r + 1$, the subspaces of $S_q^r(\triangle)$ with stable dimension introduces in [3] can be used.

In this paper we describe a quite general algorithm of local refinement of spline approximations (in particular, finite-element approximations) that does not require any local mesh refinement. To perform our local refinement, we need a nested sequence of spline spaces $S^{[0]} \subset S^{[1]} \subset \cdots \subset S^{[m]} \subset \cdots$ obtained e.g. by uniform mesh refinement and a sequence of approximation operators $Q^{[m]} : L_p(\Omega) \to S^{[m]}$, $m = 0, 1, \ldots$, with growing approximation power. Starting with $\tilde{Q}^{[0]} = Q^{[0]}$, we show how to recursively construct operators $\tilde{Q}^{[m]}$ by using $Q^{[m]}$ to enhance the approximation power of $\tilde{Q}^{[m-1]}$ in the subregions where it is needed, thus saving the computation cost in the rest of Ω. Note that uniform refinements are available for many multivariate polynomial spline constructions, such as low order polynomial splines on uniform type triangulations, tensor product splines, box splines, FVS finite elements on quadrilaterals with diagonals and Powell-Sabin 12 split finite elements. The kind of local refinement introduced here seems advantageous if a uniform type mesh can be used avoiding the expensive data structures of arbitrary triangulations needed for local mesh refinement.

2 Refinement Algorithm

Let $\Omega \subseteq \mathbb{R}^n$ be a bounded domain, and let $L_p(\Omega)$, $1 \leq p \leq \infty$, denote the usual spaces of real functions defined on Ω, with norm $\|\cdot\|_p$. For the L_p-norm on a subset G of Ω we will use the notation $\|\cdot\|_{p,G}$. Suppose

$$S^{[0]} \subset S^{[1]} \subset \cdots \subset S^{[m]} \subset \cdots \tag{2.1}$$

is a nested sequence of finite-dimensional linear subspaces of $L_p(\Omega)$.

For each $m = 0, 1, \ldots$, let the space $S^{[m]}$ be endowed with a basis $\mathcal{B}^{[m]} = \{b_j^{[m]}\}_{j \in J^{[m]}}$ and a dual basis $\Gamma^{[m]} = \{\gamma_j^{[m]}\}_{j \in J^{[m]}}$ for $(S^{[m]})^*$, where $J^{[m]}$ is a finite index set, such that

$$\gamma_j^{[m]} b_k^{[m]} = \begin{cases} 1, & \text{if } j = k, \\ 0, & \text{otherwise.} \end{cases}$$

We assume that $b_j^{[m]}$ and $\gamma_j^{[m]}$ are supported on $B_j^{[m]} \subseteq \Omega$ and $G_j^{[m]} \subseteq B_j^{[m]}$, respectively, and L_p-bounded, *i.e.*,

$$\operatorname{supp} b_j^{[m]} \subseteq B_j^{[m]}, \tag{2.2}$$

$$\|b_j^{[m]}\|_{p,B_j^{[m]}} \leq K_1, \tag{2.3}$$

$$|\gamma_j^{[m]} s| \leq K_2 \|s\|_{p,G_j^{[m]}}, \quad \text{all } s \in S^{[m]}, \tag{2.4}$$

for some constants K_1, K_2 independent of j and m.

We say that a finite set $\triangle$ of closed subsets of Ω with pairwise disjoint interiors is a *partition* of Ω if the union of these subsets covers Ω. In addition to (2.2)–(2.4), we assume that for each m there exists a finite partition $\triangle^{[m]}$ of Ω, such that each $B_j^{[m]}$ is the union of a subset of $\triangle$, and the basis $\mathcal{B}$ is *local* with respect to $\triangle^{[m]}$, *i.e.*,

$$\#\{j \in J^{[m]} \,:\, T \subset B_j^{[m]}\} \leq \ell_1, \qquad \text{all } T \in \triangle^{[m]}, \tag{2.5}$$

$$\#\{T \in \triangle^{[m]} \,:\, T \subset B_j^{[m]}\} \leq \ell_2, \qquad \text{all } j \in J^{[m]}, \tag{2.6}$$

where the constants ℓ_1, ℓ_2 do not depend on m.

For each $m = 0, 1, \ldots$, let $Q^{[m]} : L_p(\Omega) \to S^{[m]}$ be an operator (nonlinear in general). Then

$$Q^{[m]} f = \sum_{j \in J^{[m]}} (q_j^{[m]} f) b_j^{[m]},$$

where $q_j^{[m]} : L_p(\Omega) \to \mathbb{R}$, $j \in J^{[m]}$, can be calculated as

$$q_j^{[m]} f = \gamma_j^{[m]} (Q^{[m]} f). \tag{2.7}$$

Assume the functions $Q^{[0]} f, Q^{[1]} f, \ldots, Q^{[m]} f, \ldots$ are used as approximations of f, with the approximation error measured by $\|f - Q^{[m]} f\|_p$ and generally decreasing with m. It may happen that the *local approximation error*,

$$\|f - Q^{[m]} f\|_{p,T},$$

varies greatly with $T \in \triangle^{[m]}$. In this case the following modified operators $\tilde{Q}^{[m]} : L_p(\Omega) \to S^{[m]}$ provide approximations of f of the same quality as $Q^{[m]}$ at (potentially) much lower cost.

We set $\tilde{Q}^{[0]} = Q^{[0]}$. For $m \geq 1$, we assume that $\tilde{Q}^{[m-1]}$ is already defined and the local approximation quality of $\tilde{Q}^{[m-1]} f$ is satisfactory everywhere except for a subdomain

$$\Omega^{[m]} \subset \Omega,$$

where it should be improved because the error $\|f - \tilde{Q}^{[m-1]} f\|_{p,\Omega^{[m]}}$ is too high. We set

$$\begin{gathered} J^{[m]}(G) := \{ j \in J^{[m]} \; : \; \operatorname{int} G \cap B_j^{[m]} \neq \emptyset \}, \qquad G \subset \Omega, \\ \tilde{J}^{[m]} := J^{[m]}(\Omega^{[m]}), \qquad \tilde{\Omega}^{[m]} := \bigcup_{j \in \tilde{J}^{[m]}} B_j^{[m]}, \\ \hat{J}^{[m]} := J^{[m]}(\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}), \qquad \hat{\Omega}^{[m]} := \bigcup_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} G_j^{[m]}, \end{gathered}$$

and define

$$(\tilde{Q}^{[m]}f)(x) = \begin{cases} (\tilde{Q}^{[m-1]}f)(x), & \text{if } x \in \Omega \setminus \tilde{\Omega}^{[m]}, \\ \sum_{j \in \tilde{J}^{[m]}} (q_j^{[m]} f) b_j^{[m]}(x), & \text{if } x \in \Omega^{[m]}, \\ \sum_{j \in \tilde{J}^{[m]}} (q_j^{[m]} f) b_j^{[m]}(x) + \\ \qquad \sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} \gamma_j^{[m]} (\tilde{Q}^{[m-1]}f) b_j^{[m]}(x), & \text{if } x \in \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}. \end{cases} \tag{2.8}$$

Theorem 2.1 *For any $f \in L_p(\Omega)$,*

$$\tilde{Q}^{[m]} f \in S^{[m]}, \qquad m = 0, 1, \ldots .$$

Proof. We have $\tilde{Q}^{[0]} f = Q^{[0]} f \in S^{[0]}$. For $m \geq 1$, assume $\tilde{Q}^{[m-1]} f \in S^{[m-1]}$. Since

$$S^{[m-1]} \subset S^{[m]}$$

and

$$\operatorname{supp} b_j^{[m]} \subset \tilde{\Omega}^{[m]}, \qquad j \in \tilde{J}^{[m]},$$

we have for all $x \in \Omega \setminus \tilde{\Omega}^{[m]}$,

$$\begin{aligned} (\tilde{Q}^{[m]}f)(x) &= (\tilde{Q}^{[m-1]}f)(x) \\ &= \sum_{j \in J^{[m]}} \gamma_j^{[m]} (\tilde{Q}^{[m-1]}f) b_j^{[m]}(x) \\ &= \sum_{j \in J^{[m]} \setminus \tilde{J}^{[m]}} \gamma_j^{[m]} (\tilde{Q}^{[m-1]}f) b_j^{[m]}(x) \\ &= \sum_{j \in \tilde{J}^{[m]}} (q_j^{[m]} f) b_j^{[m]}(x) \; + \sum_{j \in J^{[m]} \setminus \tilde{J}^{[m]}} \gamma_j^{[m]} (\tilde{Q}^{[m-1]}f) b_j^{[m]}(x). \end{aligned}$$

Taking into account the facts that

$$\begin{aligned} &\operatorname{supp} b_j^{[m]} \cap \operatorname{int} \Omega^{[m]} = \emptyset, \qquad j \in J^{[m]} \setminus \tilde{J}^{[m]}, \\ &\operatorname{supp} b_j^{[m]} \cap \operatorname{int} (\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}) = \emptyset, \qquad j \in J^{[m]} \setminus \hat{J}^{[m]}, \end{aligned}$$

we conclude that the same formula is valid for $x \in \Omega^{[m]}$ and $x \in \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}$, respectively, *i.e.*,

$$\tilde{Q}^{[m]} f = \sum_{j \in J^{[m]}} (\tilde{q}_j^{[m]} f) b_j^{[m]},$$

with

$$\tilde{q}_j^{[m]} f = \begin{cases} q_j^{[m]} f, & \text{if } j \in \tilde{J}^{[m]}, \\ \gamma_j^{[m]}(\tilde{Q}^{[m-1]} f), & \text{if } j \in J^{[m]} \setminus \tilde{J}^{[m]}. \quad \blacksquare \end{cases} \tag{2.9}$$

As we see from (2.8), the operator $\tilde{Q}^{[m]}$ is constructed from $\tilde{Q}^{[m-1]}$ by using in addition only the coefficients $q_j^{[m]}$, $j \in \tilde{J}^{[m]}$. This will be a clear advantage over $Q^{[m]}$ if $\tilde{J}^{[m]}$ is a small part of $J^{[m]}$. Moreover, since

$$(\tilde{Q}^{[m]} f)(x) = (Q^{[m]} f)(x), \qquad x \in \Omega^{[m]},$$

$\tilde{Q}^{[m]}$ has the approximation power of $Q^{[m]}$ in the subdomain $\Omega^{[m]}$ where an improvement of the error $\tilde{Q}^{[m-1]} f - f$ is needed. On the other hand, $\tilde{Q}^{[m]} f$ coincides with $\tilde{Q}^{[m-1]} f$ on $\Omega \setminus \tilde{\Omega}^{[m]}$ and therefore retains its approximation power in this part of the domain. Next theorem shows that we can use (2.3)–(2.6) to estimate the error also in the "intermediate layer" $\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}$.

Theorem 2.2 *For any $f \in L_p(\Omega)$, $m = 0, 1, \ldots,$*

$$\begin{aligned} \|f - \tilde{Q}^{[m]} f\|_{p,\Omega^{[m]}} &= \|f - Q^{[m]} f\|_{p,\Omega^{[m]}}, \\ \|f - \tilde{Q}^{[m]} f\|_{p,\Omega \setminus \tilde{\Omega}^{[m]}} &= \|f - \tilde{Q}^{[m-1]} f\|_{p,\Omega \setminus \tilde{\Omega}^{[m]}}, \\ \|f - \tilde{Q}^{[m]} f\|_{p,\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}} &\le \|f - Q^{[m]} f\|_{p,\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}} \\ &\quad + K(\|f - Q^{[m]} f\|_{p,\hat{\Omega}^{[m]}} + \|f - \tilde{Q}^{[m-1]} f\|_{p,\hat{\Omega}^{[m]}}), \end{aligned}$$

where the constant K is independent of m.

Proof. We only have to show the third inequality. To this end it obviously suffices to prove that

$$\|Q^{[m]} f - \tilde{Q}^{[m]} f\|_{p,\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}} \le K \|Q^{[m]} f - \tilde{Q}^{[m-1]} f\|_{p,\hat{\Omega}^{[m]}}, \tag{2.10}$$

with a constant K independent of m.

Using the notation (2.7), (2.9) and the facts that $B_j^{[m]} \cap \mathrm{int}\,(\tilde{\Omega}^{[m]} \setminus \Omega^{[m]}) = \emptyset$ for all $j \notin \hat{J}^{[m]}$ and $q^{[m]} f = \tilde{q}^{[m]} f$ for all $j \in \tilde{J}^{[m]}$, we get

$$\begin{aligned}
\|Q^{[m]} f - \tilde{Q}^{[m]} f\|_{p,\tilde{\Omega}^{[m]}\setminus\Omega^{[m]}} &= \| \sum_{j \in \hat{J}^{[m]}} (q_j^{[m]} f - \tilde{q}_j^{[m]} f) b_j^{[m]} \|_{p,\tilde{\Omega}^{[m]}\setminus\Omega^{[m]}} \\
&= \| \sum_{j \in \hat{J}^{[m]}\setminus\tilde{J}^{[m]}} \gamma_j^{[m]} (Q^{[m]} f - \tilde{Q}^{[m-1]} f) b_j^{[m]} \|_{p,\tilde{\Omega}^{[m]}\setminus\Omega^{[m]}}.
\end{aligned}$$

Setting

$$\alpha_j := \gamma_j^{[m]} (Q^{[m]} f - \tilde{Q}^{[m-1]} f), \qquad j \in \hat{J}^{[m]},$$

we have for $p = \infty$,

$$\begin{aligned}
\| \sum_{j \in \hat{J}^{[m]}\setminus\tilde{J}^{[m]}} \alpha_j b_j^{[m]} \|_{\infty,\tilde{\Omega}^{[m]}\setminus\Omega^{[m]}} &= \| \sum_{j \in \hat{J}^{[m]}\setminus\tilde{J}^{[m]}} \alpha_j b_j^{[m]} \|_{\infty,\tilde{T}} \\
&= \| \sum_{j \in J^{[m]}(\tilde{T})\setminus\tilde{J}^{[m]}} \alpha_j b_j^{[m]} \|_{\infty,\tilde{T}} \\
&\le \ell_1 K_1 \max_{j \in \hat{J}^{[m]}\setminus\tilde{J}^{[m]}} |\alpha_j|,
\end{aligned}$$

where $\tilde{T} \subset \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}$ is an element of $\triangle^{[m]}$, and the constants K_1 and ℓ_1 come from (2.3) and (2.5), respectively. Now (2.4) implies

$$\max_{j \in \hat{J}^{[m]}\setminus\tilde{J}^{[m]}} |\alpha_j| \le K_2 \|Q^{[m]} f - \tilde{Q}^{[m-1]} f\|_{\infty,\hat{\Omega}^{[m]}},$$

and we arrive at the estimate

$$\|Q^{[m]} f - \tilde{Q}^{[m]} f\|_{\infty,\tilde{\Omega}^{[m]}\setminus\Omega^{[m]}} \le K_1 K_2 \ell_1 \|Q^{[m]} f - \tilde{Q}^{[m-1]} f\|_{\infty,\hat{\Omega}^{[m]}}.$$

For $p < \infty$, we have by (2.3) – (2.6),

$$
\begin{aligned}
\| \sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} \alpha_j b_j^{[m]} \|^p_{p, \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}} &= \sum_{\substack{T \in \triangle^{[m]} \\ T \subset \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}}} \| \sum_{j \in J^{[m]}(T) \setminus \tilde{J}^{[m]}} \alpha_j b_j^{[m]} \|^p_{p,T} \\
&\leq K_1^p \sum_{\substack{T \in \triangle^{[m]} \\ T \subset \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}}} \Big(\sum_{j \in J^{[m]}(T) \setminus \tilde{J}^{[m]}} |\alpha_j| \Big)^p \\
&\leq K_1^p \ell_1^{p-1} \sum_{\substack{T \in \triangle^{[m]} \\ T \subset \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}}} \sum_{j \in J^{[m]}(T) \setminus \tilde{J}^{[m]}} |\alpha_j|^p \\
&\leq K_1^p \ell_1^{p-1} \ell_2 \sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} |\alpha_j|^p \\
&\leq K_1^p \ell_1^{p-1} \ell_2 K_2^p \sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|^p_{p, G_j^{[m]}}.
\end{aligned}
$$

Since $G_j^{[m]} \subseteq B_j^{[m]}$, (2.5) implies

$$
\begin{aligned}
\sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|^p_{p, G_j^{[m]}} &= \sum_{j \in \hat{J}^{[m]} \setminus \tilde{J}^{[m]}} \sum_{\substack{T \in \triangle^{[m]} \\ T \subset G_j^{[m]}}} \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|^p_{p,T} \\
&\leq \ell_1 \sum_{\substack{T \in \triangle^{[m]} \\ T \subset \hat{\Omega}^{[m]}}} \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|^p_{p,T} \\
&= \ell_1 \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|^p_{p, \hat{\Omega}^{[m]}},
\end{aligned}
$$

and we obtain the required estimate

$$
\| Q^{[m]} f - \tilde{Q}^{[m]} f \|_{p, \tilde{\Omega}^{[m]} \setminus \Omega^{[m]}} \leq K_1 K_2 \ell_1 \ell_2^{1/p} \| Q^{[m]} f - \tilde{Q}^{[m-1]} f \|_{p, \hat{\Omega}^{[m]}}. \quad \blacksquare
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