Locally Linearly Independent Bases for Bivariate Polynomial Spline Spaces

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Abstract. Locally linearly independent bases are constructed for the spaces $\mathcal{S}_d^r(\triangle)$ of polynomial splines of degree $d \geq 3r+2$ and smoothness r defined on triangulations, as well as for their superspline subspaces.

§1. Introduction

Given a regular triangulation $\triangle$ of a set of vertices $\mathcal{V}$, let

$$\mathcal{S}_d^r(\triangle) := \{s \in C^r(\Omega) : \; s|_T \in \mathcal{P}_d \quad \text{for all triangles } T \in \triangle\},$$

where $\mathcal{P}_d$ is the space of polynomials of degree d, and Ω is the union of the triangles in $\triangle$. Suppose $\mathcal{B} := \{B_i\}_{i=1}^n$ is a basis for $\mathcal{S}_d^r(\triangle)$. Then $\mathcal{B}$ is said to be *locally linearly independent* (LLI) provided that for every $T \in \triangle$, the basis splines $\{B_i\}_{i\in\Sigma_T}$ are linearly independent on T, where

$$\Sigma_T := \{i : \; T \subseteq \text{supp}(B_i)\}. \tag{1.1}$$

Since $\mathcal{S}_d^r(\triangle)$ contains the space $\mathcal{P}_d$ of polynomials, $\mathcal{B}$ being LLI is equivalent to the condition

$$\#\Sigma_T = \dim \mathcal{P}_d = \binom{d+2}{2} \quad \text{for every} \;\; T \in \triangle. \tag{1.2}$$

For a discussion of various equivalent definitions of local linear independence, see [10,15].

Locally linearly independent bases play an important role in the theory of interpolation and almost interpolation by multivariate splines, see [14,15]. They are also of interest since an LLI basis $\mathcal{B}$ for $\mathcal{S}_d^r(\triangle)$ is a *least supported basis* in the sense that it is optimal with respect to the size of the supports of the B_i, see [7].

Locally supported bases have been constructed for the spline spaces $\mathcal{S}_d^r(\triangle)$ and their superspline subspaces in [3,4,16,17,20,24], but they are mostly not LLI,

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see Remark 8.2 below. Recently, LLI bases have been constructed for the spaces $\mathcal{S}_d^1(\triangle)$ (see [10]) and for certain superspline spaces (see [11,14]).

The main purpose of this paper is to construct LLI bases for $\mathcal{S}_d^r(\triangle)$ for all $d \geq 3r+2$, and for the entire scale of superspline spaces discussed in [17]. The paper is organized as follows. In Sect. 2 we treat the space $\mathcal{S}_d^0(\triangle)$ separately as a means of introducing some needed notation. Then in Sect. 3 we show how to use Bernstein-Bézier techniques to handle the spaces $\mathcal{S}_d^1(\triangle)$ (the results in [10] are based on nodal methods). In Sects. 4 and 5 we construct LLI bases for splines on cells. Finally, the main results on $\mathcal{S}_d^r(\triangle)$ and its superspline subspaces are established in Sect. 6 and Sect. 7, respectively. Sect. 8 is devoted to a few remarks.

§2. The space $\mathcal{S}_d^0(\triangle)$

For the sake of completeness and in order to set some notation, we briefly describe the situation for the spline space $\mathcal{S}_d^0(\triangle)$. We make use of standard Bernstein-Bézier methods as in [3,4,16,17]. Given a triangle $T = \langle v_1, v_2, v_3 \rangle$, the points

$$\xi_{ijk}^T := \frac{iv_1 + jv_2 + kv_3}{d}, \qquad i+j+k=d,$$

are called the *domain points*. Each polynomial of degree d can be written in the Bernstein-Bézier (B-) form

$$p = \sum_{i+j+k=d} c_{ijk} B_{ijk}^d,$$

where B_{ijk}^d are the Bernstein polynomials of degree d associated with the triangle.

We write $\mathcal{D}_\triangle$ for the union of all domain points where repetitions (those on the edges and at the vertices) are removed. We recall that a set $\mathcal{M} := \mathcal{D}_\triangle$ is called a *determining set for* $\mathcal{S} \subseteq \mathcal{S}_d^0(\triangle)$ provided that for all $s \in \mathcal{S}$,

$$\lambda_\xi s = 0 \text{ for all } \xi \in \mathcal{M} \quad \text{implies} \quad s \equiv 0,$$

where λ_η is the linear functional defined by

$$\lambda_\eta s = \text{the B-coefficient of } s \text{ associated with } \eta.$$

$\mathcal{M}$ is called a *minimal determining set for* $\mathcal{S}$ provided it is the smallest determining set for $\mathcal{S}$. The following lemma is well-known, see [3].

Lemma 2.1. *Suppose $\mathcal{M} \subseteq \mathcal{D}_\triangle$ is a minimal determining set for a spline space $\mathcal{S}$. For each $\xi \in \mathcal{M}$, let B_ξ be the unique spline satisfying*

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \mathcal{M}. \tag{2.1}$$

Then $\mathcal{B} := \{B_\xi\}_{\xi \in \mathcal{M}}$ is a basis for $\mathcal{S}$.

It is easy to see that $\mathcal{M} = \mathcal{D}_\triangle$ is a minimal determining set for $\mathcal{S}_d^0(\triangle)$. The corresponding dual basis splines of Lemma 2.1 have the following local supports:

$$\operatorname{supp}(B_\xi) = \begin{cases} T, & \text{if } \xi \text{ lies in the interior of } T, \\ T_1 \cup T_2, & \text{if } \xi \text{ lies in the interior of } T_1 \cap T_2, \\ \operatorname{star}(v), & \text{if } \xi \text{ lies at a vertex } v, \end{cases}$$

where $\operatorname{star}(v)$ is the union of the triangles surrounding the vertex v. A simple count shows that (1.2) holds for the basis $\mathcal{B}$, and hence it is an LLI basis for $\mathcal{S}_d^0(\triangle)$ (compare [14]).

§3. The space $\mathcal{S}_d^1(\triangle)$

In this section we show how to construct an LLI basis for $\mathcal{S}_d^1(\triangle)$ for $d \geq 5$ by choosing an appropriate minimal determining set $\mathcal{M}$. First we recall some standard notation. Given a triangle $T = \langle u, v, w \rangle$, then for any integer $0 \leq \ell \leq d$, we define

$$R_\ell^T(u) := \{\xi_{ijk}^T : \ i = d - \ell\}$$

and

$$D_\ell^T(u) := \{\xi_{ijk}^T : \ i \geq d - \ell\}$$

with similar definitions for the vertices v and w. Associated with the edge $e = \langle v, w \rangle$, we define

$$E^T(e) = \{\xi_{ijk}^T : \ i \leq 1\} \setminus [D_2^T(v) \cup D_2^T(w)],$$

with similar definitions for the other two edges of T. Finally, we set

$$C^T := \{\xi_{ijk}^T : \ i, j, k \geq 2\}.$$

Given a vertex v of $\triangle$, we define the *ring* $R_\ell(v)$ of radius ℓ around v and the *disk* $D_\ell(v)$ of radius ℓ around v to be the sets

$$R_\ell(v) := \bigcup \{R_\ell^T(v) : \ T \text{ is a triangle with vertex at } v\},$$

$$D_\ell(v) := \bigcup_{i=1}^{\ell} R_i(v).$$

We also need the well-known concepts of degenerate edges and singular vertices. Suppose v is a vertex of $\triangle$ which is connected to vertices $v_1, \ldots, v_n$ in counterclockwise order. If v is an interior vertex we define $v_{n+1} = v_1$ and $v_0 = v_n$ for convenience. Then an edge $\langle v, v_i \rangle$ is called *degenerate at* v provided that the two edges $\langle v, v_{i-1} \rangle$ and $\langle v, v_{i+1} \rangle$ are collinear. An interior vertex v is called *singular* if $n = 4$ and all four edges are degenerate at v.

Theorem 3.1. *Suppose $d \geq 5$, and let $\mathcal{M}$ be the following set:*

1) *For each triangle T, choose the $\binom{d-4}{2}$ domain points in C^T.*
2) *For each edge e, pick one triangle T which shares the edge e, and choose the domain points in $E^T(e)$.*
3) *For each interior vertex v of $\triangle$ connected to vertices $v_1, \ldots, v_n$ in counterclockwise order,*
   a) *choose any three non-collinear points in $D_1(v)$,*
   b) *for each $i = 1, \ldots, n$, if $\langle v, v_i \rangle$ is nondegenerate at v, choose the point on $R_2(v)$ in the interior of the triangle $\langle v, v_i, v_{i+1} \rangle$. Otherwise, choose the point $R_2(v) \cap \langle v, v_i \rangle$.*
   c) *if v is a singular vertex, add the point on $R_2(v)$ in the interior of the triangle $\langle v, v_1, v_2 \rangle$.*
4) *For each boundary vertex v of $\triangle$ connected to vertices $v_1, \ldots, v_n$ in counterclockwise order,*
   a) *choose any three non-collinear points in $D_1(v)$,*
   b) *for each $i = 1, \ldots, n-1$, if $\langle v, v_i \rangle$ is nondegenerate at v, choose the point on $R_2(v)$ in the interior of the triangle $\langle v, v_i, v_{i+1} \rangle$. Otherwise, choose the point $R_2(v) \cap \langle v, v_i \rangle$.*
   c) *add the two points $R_2(v) \cap \langle v, v_1 \rangle$ and $R_2(v) \cap \langle v, v_n \rangle$.*

Then $\mathcal{M}$ is a minimal determining set for $\mathcal{S}_d^1(\triangle)$.

Proof: It is straightforward to verify that $\mathcal{M}$ is a determining set for $\mathcal{S}_d^1(\triangle)$ and that for each $\xi \in \mathcal{M}$ there exists a unique spline B_ξ satisfying (2.1). Clearly these *dual splines* are linearly independent, and it follows that $\mathcal{M}$ is actually a minimal determining set. □

It is easy to see that each of the basis functions in Theorem 3.1 has support which is contained in the star of a vertex. This approach to definining a minimal determining set $\mathcal{M}$ for $\mathcal{S}_d^1(\triangle)$ was used already in [3], but without specifying explicitly the determining sets in $D_2(v)$ described in steps 3 and 4. These steps have to be done carefully in order to get an LLI basis. The construction here is just a translation to Bézier form of the nodal construction given in [10].

Theorem 3.2. *Let $\mathcal{M}$ be the set constructed in Theorem 3.1. Then the set of dual splines $\mathcal{B} := \{B_\xi\}_{\xi \in \mathcal{M}}$ forms an LLI basis for $\mathcal{S}_d^1(\triangle)$.*

Proof: Given a triangle T, let $\nu_i(T)$ denote the number of splines B_ξ whose supports overlap T which correspond to domain points chosen in step i of the algorithm. Since all of the splines constructed in step 1 overlap T,

$$\nu_1(T) = \binom{d-4}{2}.$$

In step 2 we get a spline B_ξ which overlaps T if and only if e is one of the three edges of T. Since the cardinality of $E^T(e)$ is $2d-9$, this gives

$$\nu_2(T) = 3(2d-9).$$

Finally, it is also easy to see that the choices in steps 3 and 4 lead to

$$\nu_3(T) + \nu_4(T) = 3 \times 6 = 18, \tag{3.1}$$

and it follows that

$$\#\Sigma_T = \nu_1(T) + \nu_2(T) + \nu_3(T) + \nu_4(T) = (d^2 + 3d + 2)/2,$$

which is the dimension of $\mathcal{P}_d$. $\square$

§4. The space $\mathcal{S}_\mu^r(\triangle_v)$ for an interior cell $\triangle_v$

In this section we construct an LLI basis for the space of splines $\mathcal{S}_\mu^r(\triangle_v)$ defined on an *interior cell* $\triangle_v$, *i.e.*, a set of triangles surrounding a single interior vertex v. Here μ is an arbitrary integer such that $\mu \geq r+1$. Suppose that the vertices connected to v are $v_1, \ldots, v_n$ in counterclockwise order. Let $T^{[i]} = \langle v, v_i, v_{i+1} \rangle$, for $i = 1, \ldots, n$, where we identify $v_{n+1} = v_1$.

To construct an LLI basis for $\mathcal{S}_\mu^r(\triangle_v)$, we first decompose it into a direct sum of certain subspaces. Let

$$\mathcal{V}_{r,m} := \{g \in \mathcal{S}_m^r(\triangle_v) : \ D_x^\alpha D_y^\beta g(v) = 0, \quad 0 \leq \alpha + \beta \leq m-1\}, \tag{4.1}$$

for $r+1 \leq m \leq \mu$. Then clearly

$$\mathcal{S}_\mu^r(\triangle_v) = \mathcal{P}_r \oplus \mathcal{V}_{r,r+1} \oplus \cdots \oplus \mathcal{V}_{r,\mu}.$$

Lemma 4.1. *Suppose that for each $1 \leq j \leq \mu - r$, the set $\{g_{1,v}^{[j]}, \ldots, g_{n_j,v}^{[j]}\}$ is an LLI basis for $\mathcal{V}_{r,r+j}$. Then*

$$\{x^\alpha y^\beta : \ 0 \leq \alpha + \beta \leq r\} \cup \{g_{1,v}^{[1]}, \ldots, g_{n_1,v}^{[1]}\} \cup \cdots \cup \{g_{1,v}^{[\mu-r]}, \ldots, g_{n_{\mu-r},v}^{[\mu-r]}\} \tag{4.2}$$

is an LLI basis for $\mathcal{S}_\mu^r(\triangle_v)$.

Proof: Let

$$\mathcal{H}_{r+j} := \operatorname{span}\{x^\alpha y^\beta : \ \alpha + \beta = r + j\}$$

be the $(r+j+1)$-dimensional space of homogeneous polynomials of degree $r+j$. It is obvious that

$$\mathcal{H}_{r+j} \subseteq \mathcal{V}_{r,r+j}, \qquad j = 1, \ldots, \mu - r.$$

Now fix a triangle $T \in \triangle_v$. Then for every $g \in \mathcal{V}_{r,r+j}$, we have $g|_T \in \mathcal{H}_{r+j}$, which implies

$$\dim \mathcal{V}_{r,r+j}|_T = r + j + 1, \qquad j = 1, \ldots, \mu - r.$$

Since $\{g_{1,v}^{[j]}, \ldots, g_{n_j,v}^{[j]}\}$ are locally linearly independent, it follows that

$$\#\{k : \ T \subseteq \operatorname{supp}(g_{k,v}^{[j]})\} = \dim \mathcal{V}_{r,r+j}|_T = r + j + 1.$$

Therefore, the total number of basis functions in (4.2) supported on T is

$$\binom{r+2}{2} + \sum_{j=1}^{\mu-r} (r + j + 1) = \binom{\mu+2}{2} = \dim \mathcal{P}_\mu,$$

which shows (1.2) and proves our claim. $\square$

We turn now to the task of constructing LLI bases for the spaces $\mathcal{V}_{r,r+j}$ for $j = 1, \ldots, \mu - r$. We make use of the cofactor approach used in [21–23]. Without loss of generality, we may assume that $v = (0, 0)$ and the cell is rotated so that all of the coordinates (x_i, y_i) of the points v_i are nonzero. Let $y + \alpha_i x = 0$ be the equation of the i-th edge e_i attached to v, where $\alpha_i = -y_i/x_i$. Then every spline $g \in S_\mu^r(\triangle_v)$ can be written in the form

$$g|_{T^{[\ell]}} = \sum_{j=0}^{\mu} \sum_{k=0}^{j} a_{jk}^{\mathcal{P}} y^k x^{j-k} + \sum_{i=1}^{\ell} \sum_{j=1}^{\mu-r} \sum_{k=1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k}, \qquad \ell = 1, \ldots, n, \tag{4.3}$$

with

$$\sum_{i=1}^{n} \sum_{j=1}^{\mu-r} \sum_{k=1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k} \equiv 0, \tag{4.4}$$

see [21].

For later use we define some linear functionals which can be used to pick off the coefficients in (4.3). Let

$$\chi_{jk}^{\mathcal{P}} g := \tfrac{1}{k!(j-k)!} D_y^k D_x^{j-k} \left(g|_{T^{[n]}} \right)(v),$$

$$\chi_{jk}^{[i]} g := \beta_{jk}^{[i]} \left(D_{e_i^\perp}^{r+k} D_x^{j-k} \left(g|_{T^{[i]}} \right)(v) - D_{e_i^\perp}^{r+k} D_x^{j-k} \left(g|_{T^{[i-1]}} \right)(v) \right), \ i = 1, \ldots, n,$$

where $T^{[0]}$ is identified with $T^{[n]}$,

$$\beta_{jk}^{[i]} := \frac{(1 + \alpha_i^2)^{(r+k)/2}}{((r+k)!)^2} \left(\sum_{\nu=0}^{r+k} \frac{(r + j - \nu)!}{(\nu!)^2 (j - k + \nu)!} \alpha_i^{2\nu} \right)^{-1},$$

and $D_{e_i^\perp}$ denotes the derivative in the normal direction to e_i, *i.e.*,

$$D_{e_i^\perp} := (1 + \alpha_i^2)^{-1/2} (D_y + \alpha_i D_x).$$

Then for any spline $g \in \mathcal{S}^r_\mu(\triangle_v)$, $a^{\mathcal{P}}_{jk} = \chi^{\mathcal{P}}_{jk} g$ and $a^{[i]}_{jk} = \chi^{[i]}_{jk} g$ for $i = 1, \dots, n$. By Theorem 2.2 in [23],

$$n_j := \dim \mathcal{V}_{r,r+j} = nj + (r + j + 1 - je)_+, \qquad j = 1, \dots, \mu - r,$$

where e is the number of edges attached to v with *different* slopes. We distinguish three cases.

Case 1: Suppose $r + j + 1 \geq je$. Then $n_j = r + j + 1 + (n - e)j$. If $e = n$, *i.e.*, all edges attached to v have different slopes, then $\mathcal{V}_{r,r+j} = \mathcal{H}_{r+j}$, and any basis for the space of homogeneous polynomials $\mathcal{H}_{r+j}$ is locally linearly independent and can be used in our construction.

Assuming that $n - e \geq 1$, we choose $n - e$ indices $\ell_1, \dots, \ell_{n-e} \in \{1, \dots, n\}$ such that the associated edges $e_{\ell_i} = \langle v, v_{\ell_i} \rangle$, $i = 1, \dots, n - e$, are pairwise noncollinear, but each of them has a collinear counterpart among e_ℓ, $\ell \notin \{\ell_1, \dots, \ell_{n-e}\}$. Then the truncated powers

$$x^{j-1}(y + \alpha_{\ell_i} x)^{r+1}_+, \dots, (y + \alpha_{\ell_i} x)^{r+j}_+, \;\; x^{j-1}(y + \alpha_{\ell_i} x)^{r+1}_-, \dots, (y + \alpha_{\ell_i} x)^{r+j}_- \quad (4.5)$$

lie in $\mathcal{V}_{r,r+j}$ for all $i = 1, \dots, n - e$. We want to extend this set of $2(n - e)j$ functions to an LLI basis for $\mathcal{V}_{r,r+j}$. To this end we first show that the following $(n - e)j$ homogeneous polynomials

$$x^{j-1}(y + \alpha_{\ell_i} x)^{r+1}, \dots, (y + \alpha_{\ell_i} x)^{r+j}, \qquad i = 1, \dots, n - e, \qquad (4.6)$$

are linearly independent. Indeed, the identity

$$\sum_{i=1}^{n-e} \sum_{k=1}^{j} c_{ik} x^{j-k} (y + \alpha_{\ell_i} x)^{r+k} \equiv 0$$

is equivalent to a system of $r + j + 1$ linear equations in $(n - e)j$ unknowns

$$\tilde{A}_j \,(c_{1j}, \dots, c_{11}, \dots, c_{n-e,j}, \dots, c_{n-e,1})^T = 0,$$

where $\tilde{A}_j = (A^{[\ell_1]}_j, \dots, A^{[\ell_{n-e}]}_j)$, and $A^{[\ell_i]}_j$ is given by

$$A^{[\ell]}_j = \begin{pmatrix} 1 & 0 & & \\ \binom{r+j}{1}\alpha_\ell & 1 & & \\ \binom{r+j}{2}\alpha_\ell^2 & \binom{r+j-1}{1}\alpha_\ell & \ddots & \\ \cdot & \cdot & \ddots & 1 \\ \cdot & \cdot & \cdot & \binom{r+1}{1}\alpha_\ell \\ \vdots & \vdots & \vdots & \vdots \\ \binom{r+j}{r+j}\alpha_\ell^{r+j} & \binom{r+j-1}{r+j-1}\alpha_\ell^{r+j-1} & \cdots & \binom{r+1}{r+1}\alpha_\ell^{r+1} \end{pmatrix}. \qquad (4.7)$$

Since $n-e \le e$, we have

$$r+j+1 \geq (n-e)j,$$

and hence

$$\operatorname{rank} \tilde{A}_j = \min\{r+j+1, (n-e)j\} = (n-e)j.$$

It follows that the above system has only the trivial solution

$$c_{ik} = 0, \qquad i = 1, \ldots, n-e, \quad k = 1, \ldots, j,$$

which shows that the homogeneous polynomials (4.6) are linearly independent. Therefore, there exist $r+j+1-(n-e)j$ homogeneous polynomials

$$H_1, \ldots, H_{r+j+1-(n-e)j} \in \mathcal{H}_{r+j} \tag{4.8}$$

which extend the set (4.6) to a basis of $\mathcal{H}_{r+j}$. We claim that the union of the two sets (4.5) and (4.8) is a locally linearly independent basis for $\mathcal{V}_{r,r+j}$. Since for every $k = 1, \ldots, j$ and $i = 1, \ldots, n-e$ only one of the two truncated powers

$$x^{j-k}(y+\alpha_{\ell_i}x)_+^{r+k} \quad \text{and} \quad x^{j-k}(y+\alpha_{\ell_i}x)_-^{r+k}$$

is supported on a given triangle $T \in \triangle_v$, the number of functions (4.5) and (4.8) supported on T is $r+j+1 = \dim \mathcal{H}_{r+j}$. The statement now follows from the simple facts that the total number of functions in (4.5) and (4.8) is n_j, and that their span contains $\mathcal{H}_{r+j}$.

Case 2: Suppose $j(n-1) \le r+j+1 < je$. Then in particular $n-1 < e$, which implies $n = e$, *i.e.*, all edges e_i have different slopes. It follows that $n_j = \dim \mathcal{V}_{r,r+j} = nj$. Let $\omega := nj - (r+j+2)$. Note that $0 \le \omega < j$.

For each $1 \le i \le n$, we first construct $\omega+1$ splines $g_{i,0}, \ldots, g_{i,\omega}$ in $\mathcal{V}_{r,r+j}$ which vanish identically on exactly the one triangle $T^{[i-1]}$ in $\triangle_v$. Given $0 \le \nu \le \omega$, we construct $g_{i,\nu}$ by demanding that

$$\begin{aligned} &g_{i\nu}|_{T^{[i-1]}} \equiv 0, \\ &\chi_{jk}^{[i]} g_{i\nu} = 0, \quad k = 1, \ldots, \nu, \\ &\chi_{jk}^{[i-1]} g_{i\nu} = 0, \quad k = 1, \ldots, \omega - \nu. \end{aligned} \tag{4.9}$$

Then for $\ell = i, \ldots, i+n-2$,

$$g_{i\nu}|_{T^{[\ell]}}(x,y) = \sum_{k=\nu+1}^{j} a_{jk}^{[i]}(y+\alpha_i x)^{r+k} x^{j-k} + \sum_{\eta=i+1}^{\ell} \sum_{k=1}^{j} a_{jk}^{[\eta]}(y+\alpha_\eta x)^{r+k} x^{j-k}$$

with coefficients satisfying

$$\begin{aligned} \sum_{k=\nu+1}^{j} a_{jk}^{[i]}(y+\alpha_i x)^{r+k} x^{j-k} &+ \sum_{\eta=i+1}^{i+n-2} \sum_{k=1}^{j} a_{jk}^{[\eta]}(y+\alpha_\eta x)^{r+k} x^{j-k} \\ &+ \sum_{k=\omega-\nu+1}^{j} a_{jk}^{[i-1]}(y+\alpha_{i-1} x)^{r+k} x^{j-k} \equiv 0. \end{aligned}$$

This last condition is equivalent to the linear system

$$\tilde{A}_{i\nu}\,(a_j^{[i,\nu]}, a_j^{[i+1]}, \ldots, a_j^{[i+n-2]}, a_j^{[i-1,\omega-\nu]})^T = 0, \tag{4.10}$$

where

$$a_j^{[\eta]} := (a_{jj}^{[\eta]}, \ldots, a_{j1}^{[\eta]}), \quad a_j^{[\eta,\beta]} := (a_{jj}^{[\eta]}, \ldots, a_{j,\beta+1}^{[\eta]}),$$

$$\tilde{A}_{i\nu} := (A_j^{[i,\nu]} A_j^{[i+1]} \cdots A_j^{[i+n-2]} A_j^{[i-1,\omega-\nu]}),$$

$A_j^{[\eta]}$ is defined as in (4.7), and $A_j^{[\eta,\beta]}$ denotes the matrix obtained from $A_j^{[\eta]}$ by removing the last β columns. The linear system (4.10) has $r+j+1$ equations and $r+j+2$ unknowns. As in [23] it can be shown that its matrix $\tilde{A}_{i\nu}$ has full rank, which implies that $g_{i,\nu}$ satisfying (4.9) exists (and is unique up to a constant factor).

We now show that the functions $g_{i\nu}$ are linearly independent. To this end it suffices to show that for each $\ell = 1, \ldots, n$, $\{g_{i\nu} :\ i \neq \ell+1\}$ are linearly independent on $T^{[\ell]}$. Without loss of generality we only consider the case $\ell = n$. Let

$$\phi_{i\nu}(x,y) = \begin{cases} g_{i\nu}(x,y), & \text{if } (x,y) \in T^{[i]} \cup \cdots \cup T^{[n]}, \\ 0, & \text{otherwise.} \end{cases}$$

Since $\phi_{i\nu}|_{T^{[n]}} = g_{i\nu}|_{T^{[n]}}$ for all i and ν, our assertion will be established if we show that $\{\phi_{i\nu}\}_{i=2,\nu=0}^{n,\omega}$ are linearly independent on $T^{[n]}$. Assuming

$$g := \sum_{i=2}^{n} \sum_{\nu=0}^{\omega} c_{i\nu}\phi_{i\nu} \equiv 0 \quad \text{on } T^{[n]},$$

we obviously have $g \in \mathcal{V}_{r,r+j}$. Suppose not all $c_{i\nu}$ are zero, and let c_{i_0,ν_0} be the first nonzero one in the sequence $c_{20}, \ldots, c_{2\omega}, \ldots, c_{n0}, \ldots, c_{n\omega}$. Then

$$g|_{T^{[n]}}(x,y) = \sum_{k=\nu_0+1}^{j} a_{jk}^{[i_0]}(y+\alpha_{i_0}x)^{r+k}x^{j-k} + \sum_{\eta=i_0+1}^{n} \sum_{k=1}^{j} a_{jk}^{[\eta]}(y+\alpha_\eta x)^{r+k}x^{j-k},$$

with $a_{j,\nu_0+1}^{[i_0]} \neq 0$. Now $g|_{T^{[n]}} \equiv 0$ implies

$$(A_j^{[i_0,\nu_0]} A_j^{[i_0+1]} \cdots A_j^{[n]})\,(a_j^{[i_0,\nu_0]}, a_j^{[i_0+1]}, \ldots, a_j^{[n]})^T = 0.$$

Since the matrix $(A_j^{[i_0,\nu_0]} A_j^{[i_0+1]} \cdots A_j^{[n]})$ has full rank and the number of unknowns $(n-i_0)j + j - \nu_0 \leq j(n-1)$ does not exceed the number of equations $r+j+1$, we conclude that the linear system has only trivial solution, in particular $a_{j,\nu_0+1}^{[i_0]} = 0$, a contradiction which implies that all of $c_{i\nu}$'s must be zero.

To complete the construction of an LLI basis for $\mathcal{V}_{r,r+j}$, we can now take any set of splines $\tilde{g}_1, \ldots, \tilde{g}_{n(j-\omega-1)}$ in $\mathcal{V}_{r,r+j}$ such that

$$\{g_k\}_{k=1}^{n_j} := \{g_{i\nu}\}_{i=1,\nu=0}^{n,\omega} \cup \{\tilde{g}_i\}_{i=1}^{n(j-\omega-1)}$$

form a basis for $\mathcal{V}_{r,r+j}$. This basis is locally linearly independent since for each $1 \le \ell \le n$, the number of basis functions supported on the triangle $T^{[\ell]}$ is

$$(n-1)(\omega+1) + n(j-\omega-1) = r + j + 1.$$

This completes the proof in Case 2.

Case 3: Suppose $r + j + 1 < \min\{j(n-1), je\}$. In this case $n_j = \dim \mathcal{V}_{r,r+j} = nj$. We now define basis functions $\{g_{i\nu}\}_{i=1,\nu=0}^{n,j-1}$ according to the following rule. Given i and ν, let $i < i_1 < \cdots < i_e \le i + n$ be such that the associated edges $e_{i_1}, \ldots, e_{i_e}$ are pairwise noncollinear, with $i_e < i + n$ if e_i has a collinear counterpart. Let furthermore (i_η, q) denote the $(r+j+2)$-th element of the sequence

$$(i, \nu+1), \ldots, (i, j), (i_1, 1), \ldots, (i_1, j), \ldots, (i_e, 1), \ldots, (i_e, j).$$

Since $r + j + 1 < \min\{j(n-1), je\}$, it is easy to see that $i_\eta < i + n$. If for some $\zeta \le \eta$ the edge e_{i_ζ} is collinear with e_i, then $g_{i\nu}$ coincides with one of two truncated power functions

$$x^{j-\nu-1}(y + \alpha_i x)_+^{r+\nu+1} \quad \text{or} \quad x^{j-\nu-1}(y + \alpha_i x)_-^{r+\nu+1},$$

namely, the one with support $T^{[i]} \cup \cdots \cup T^{[i_\zeta - 1]}$. Otherwise, $g_{i\nu}|_{T^{[\ell]}} \equiv 0$ for $\ell = i_\eta, i_\eta + 1, \ldots, i + n - 1$, and

$$g_{i\nu}|_{T^{[\ell]}}(x,y) = \sum_{k=\nu+1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k} + \sum_{\gamma:\ i_\gamma \le \ell} \ \sum_{k=1}^{j} a_{jk}^{[i_\gamma]} (y + \alpha_{i_\gamma} x)^{r+k} x^{j-k}$$

for $\ell = i, \ldots, i_\eta - 1$, with

$$\sum_{k=\nu+1}^{j} a_{jk}^{[i]} (y + \alpha_i x)^{r+k} x^{j-k} + \sum_{\gamma=1}^{\eta-1} \sum_{k=1}^{j} a_{jk}^{[i_\gamma]} (y + \alpha_{i_\gamma} x)^{r+k} x^{j-k}$$
$$+ \sum_{k=j-q+1}^{j} a_{jk}^{[i_\eta]} (y + \alpha_{i_\eta} x)^{r+k} x^{j-k} \equiv 0.$$

In the same way as in Case 2 it can be shown that $g_{i\nu}$ are uniquely defined up to a constant factor and are linearly independent. It only remains to show that for each $1 \le \ell \le n$, the number of basis functions supported on the triangle $T^{[\ell]}$ is $r + j + 1$. Let $\ell + 1 - n \le i_1 < \cdots < i_e \le \ell$ be such that the associated edges

$e_{i_1}, \ldots, e_{i_e}$ are pairwise noncollinear, and for each $\gamma = 1, \ldots, e$, if e_{i_γ} has a collinear couterpart $e_{i'}$, with $\ell + 1 - n \le i' \le \ell$, then $i' < i_\gamma$. It is not difficult to see that, according to the above construction, the basis splines whose support includes $T^{[\ell]}$ are exactly the first $r + j + 1$ elements of the sequence

$$g_{i_e, j-1}, \ldots, g_{i_e, 0}, \ldots, g_{i_1, j-1}, \ldots, g_{i_1, 0},$$

and our assertion follows. Furthermore, let $g_{i_p, q}$ be the $(r + j + 1)$-th element of this sequence. For the purpose of an application in the next section, we note that the set of basis functions whose support includes at least one of the consecutive triangles

$$T^{[\ell]}, T^{[\ell+1]}, \ldots, T^{[\ell+\eta]}, \quad \text{with} \quad \ell + \eta \le i_p - 1 + n,$$

consists of the above mentioned $r + j + 1$ basis functions supported on $T^{[\ell]}$ and an additional ηj functions

$$g_{\ell+1,0}, \ldots, g_{\ell+1,j-1}, \ldots, g_{\ell+\eta,0}, \ldots, g_{\ell+\eta,j-1}.$$

§5. The space $\mathcal{S}_\mu^r(\triangle_v)$ for a boundary cell $\triangle_v$

Suppose $\triangle_v$ is a *boundary cell*, *i.e.*, a collection of triangles sharing a boundary vertex v. Using Lemma 4.1 as before, to construct an LLI basis for $\mathcal{S}_\mu^r(\triangle_v)$, it suffices to find LLI bases for the spaces $\mathcal{V}_{r,r+j}$ for $j = 1, \ldots, \mu - r$. It is easy to see that

$$n_j := \dim \mathcal{V}_{r,r+j} = r + j + 1 + (n-2)j.$$

We extend the boundary cell

$$\triangle_v = \{T^{[1]}, \ldots, T^{[n-1]}\}$$

to an interior cell

$$\tilde{\triangle}_v = \{T^{[1]}, \ldots, T^{[n-1]}, T^{[n]}, \ldots, T^{[n+p]}\}, \qquad p \ge 0,$$

such that v is the only interior vertex of $\tilde{\triangle}_v$. Suppose p is sufficiently large to ensure that

$$r + j + 1 < \min\{j(n + p - 1), j\tilde{e}\},$$

where $\tilde{e}$ is the number of edges of $\tilde{\triangle}_v$ attached to v with different slopes. Then an LLI basis for

$$\tilde{\mathcal{V}}_{r,r+j} := \{g \in \mathcal{S}_{r+j}^r(\tilde{\triangle}_v) : \ D_x^\alpha D_y^\beta g(v) = 0, \quad 0 \le \alpha + \beta \le r + j - 1\}$$

can be constructed by using the algorithm of Case 3 of the previous section. Moreover, the remark at the end of the proof of Case 3 shows that, by enlarging p if needed, we may assume that there are exactly $r + j + 1 + (n-2)j$ basis functions whose support includes at least one of the triangles $T^{[1]}, \ldots, T^{[n-1]}$. It then follows that the restrictions of these functions to the set $T^{[1]} \cup \cdots \cup T^{[n-1]}$ form a locally linearly independent basis for $\mathcal{V}_{r,r+j}$.

§6. The space $\mathcal{S}_d^r(\triangle)$

To construct an LLI basis for the space $\mathcal{S}_d^r(\triangle)$ for $d \ge 3r+2$, we first define a convenient determining set following the ideas of [17] and the notation of [19]. Let

$$\mu = r + \kappa, \qquad \kappa := \left\lfloor \frac{r+1}{2} \right\rfloor. \tag{6.1}$$

Given a triangle $T := \langle u, v, w \rangle$, let ξ_{ijk}^T be the domain points of $\mathcal{S}_d^r(\triangle)$ which lie in T. Let

$$C^T := \{\xi_{ijk}^T : \ i > r, \ j > r, \ k > r\}.$$

Associated with u, let

$$A^T(u) := \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{d-2r+i-1,r-j,r-i+j+1}^T\},$$

with similar definitions for the other two vertices of T. Associated with the edge $e := \langle u, v \rangle$, we define

$$\begin{aligned}
F^T(e) &:= \{\xi_{ijk}^T : \ k \le r\} \\
G_L^T(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{d-2r+i-1,r+1+j,r-i-j}^T\} \\
G_R^T(e) &:= \bigcup_{i=1}^{\lfloor \frac{r}{2} \rfloor} \bigcup_{j=0}^{i-1} \{\xi_{r+1+j,d-2r+i-1,r-i-j}^T\} \\
E^T(e) &:= F^T(e) \setminus \Big[D_\mu^T(u) \cup D_\mu^T(v) \cup A^T(u) \cup A^T(v) \cup G_L^T(e) \cup G_R^T(e)\Big],
\end{aligned} \tag{6.2}$$

with similar definitions for the other two edges of T. We now describe a determining set for $\mathcal{S}_d^r(\triangle)$.

Lemma 6.1. *Let $\mathcal{M} := \mathcal{M}_0 \cup \bigcup_v D_\mu(v)$, where $\mathcal{M}_0$ is the following set of domain points:*

1) *for each triangle T, choose the set C^T.*
2) *for each edge e, pick a triangle T sharing the edge e, and choose the set $E^T(e)$.*
3) *for each triangle $T := \langle u, v, w \rangle$, choose all three of the sets $A^T(u)$, $A^T(v)$ and $A^T(w)$.*
4) *if the edge $e := \langle v, w \rangle$ of $T := \langle u, v, w \rangle$ is degenerate at v, replace the set $A^T(v)$ by the set $G_L^T(e)$.*
5) *for each triangle T with an edge e on the boundary of Ω, add the sets $G_L^T(e)$ and $G_R^T(e)$.*

6) *for each singular vertex v, add a set of the form $A^T(v)$.*

Then $\mathcal{M}$ is a determining set for $\mathcal{S}_d^r(\triangle)$.

Proof: It is straightforward to check (cf. [17,19]) that if s is a spline $\mathcal{S}_d^r(\triangle)$ whose coefficients corresponding to the points in $\mathcal{M}$ are zero, then $s \equiv 0$. $\square$

Note that $\mathcal{M}_0$ is part of a standard minimal determining set for $\mathcal{S}_d^r(\triangle)$ (cf. [17]), but $\mathcal{M}$ is not a minimal determining set for $\mathcal{S}_d^r(\triangle)$, since it contains too many points in the disks $D_\mu(v)$. For each vertex $v \in \triangle$, let

$$m_v := \dim \mathcal{S}_\mu^r(\triangle_v).$$

We now construct m_v splines $\{B_{i,v}\}_{i=1}^{m_v}$ in $\mathcal{S}_d^r(\triangle)$ whose supports are included in $\mathrm{star}(v)$, and which satisfy

$$\lambda_\eta B_{i,v} = 0, \qquad \text{all } \eta \in \mathcal{M}_v := \mathcal{M} \setminus D_\mu(v).$$

This can be done by applying the following lemma to the LLI basis $\{g_{i,v}\}_{i=1}^{m_v}$ for the space $\mathcal{S}_\mu^r(\triangle_v)$ constructed in Sects. 4 and 5.

Theorem 6.2. *Let v be a vertex of a triangulation $\triangle$, and let $\triangle_v$ be the triangulation of* $\mathrm{star}(v)$. *Then for any $g \in \mathcal{S}_\mu^r(\triangle_v)$, there exists an associated spline $s \in \mathcal{S}_d^r(\triangle)$ with* $\mathrm{supp}(s) \subseteq \mathrm{star}(v)$ *such that*

$$\lambda_\eta s = 0, \qquad \textit{all } \eta \in \mathcal{M}_v. \tag{6.3}$$

Moreover, if g vanishes on any triangle of $\mathrm{star}(v)$, *then so does s.*

Proof: To define s, we first identify the domain points of $\mathcal{S}_\mu^r(\triangle_v)$ with the domain points associated with $\mathcal{S}_d^r(\triangle)$ and lying in $D_\mu(v)$. Then we choose the coefficients of s corresponding to these domain points to be equal to the associated coefficients of g. Then for all $\eta \in \mathcal{M}_v$, we set the coefficients $\lambda_\eta s = 0$. This assures that (6.3) holds. Then all remaining coefficients of s are computed from smoothness conditions in the usual way (cf. [17]). In particular, the coefficients in the rings of radius $\mu+1$, $\mu+2$, etc., are successively computed for all vertices of the triangulation, until all coefficients in $2r$-disks are known. After this the sets $E^T(e) \setminus \left[D_{2r}^T(u) \cup D_{2r}^T(w)\right]$ are processed for all edges $e = \langle u, w \rangle$. This procedure obviously implies that

$$\lambda_\xi s = 0, \qquad \text{all } \xi \notin D_{d-r}(v),$$

and thus $\mathrm{supp}(s) \subseteq \mathrm{star}(v)$. We now show that for every triangle T in $\mathrm{star}(v)$, $s|_T \equiv 0$ whenever $g|_T \equiv 0$. Suppose $g|_T \equiv 0$. Then

$$\lambda_\xi s = 0, \qquad \text{all } \xi \in D_\mu(v) \cap T.$$

Because of the choice of $\mathcal{M}_0$ in Lemma 6.1, it is easy to see that the computation of the remaining coefficients in T does not involve any nonzero coefficients in $D_\mu(v) \setminus T$, which implies that $s|_T \equiv 0$. $\square$

Theorem 6.3. *For each vertex $v \in \triangle$, let $\{B_{i,v}\}_{i=1}^{m_v}$ be the set of splines in $\mathcal{S}_d^r(\triangle)$ constructed from the $\{g_{i,v}\}_{i=1}^{m_v}$ of Sects. 4 and 5 using Theorem 6.2. In addition, for each $\xi \in \mathcal{M}_0$, let B_ξ be the spline in $\mathcal{S}_d^r(\triangle)$ which satisfies*

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \textit{all } \eta \in \mathcal{M}. \tag{6.4}$$

Then the set

$$\mathcal{B} := \{B_\xi\}_{\xi \in \mathcal{M}_0} \cup \bigcup_v \{B_{i,v}\}_{i=1}^{m_v}$$

forms an LLI basis for $\mathcal{S}_d^r(\triangle)$.

Proof: Using the linear independence of the $g_{i,v}$ on the cells $\triangle_v$ and (6.4), it is not hard to see that the splines in $\mathcal{B}$ are linearly independent. Since the number of splines in $\mathcal{B}$ is the same as the number of basis functions constructed in [17], it follows that $\mathcal{B}$ is a basis for $\mathcal{S}_d^r(\triangle)$. To show that it is an LLI basis, we now verify (1.2).

Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, we first examine the number of splines B_ξ with $\xi \in \mathcal{M}_0$ whose supports overlap T. By the support properties of these basis splines, it is clear that

$$\#\{\xi \in \mathcal{M}_0 : \ T \subseteq \mathrm{supp}(B_\xi)\} = \ \#C \ + \ 3\,\#E \ + \ 9\,\#A,$$

where $\#C, \#E, \#A$ are the cardinalities of the sets of the form C^T, E^T and A^T, respectively. This is precisely the number of domain points in T which lie outside of the disks $D_\mu^T(v_j)$ for $j = 1, 2, 3$. Now by the local linear independence of the $\{g_{i,v_j}\}_{i=1}^{m_{v_j}}$, it follows that

$$\#\{i : \ T \subseteq \mathrm{supp}(B_{i,v_j})\} = \#\{i : \ T \subseteq \mathrm{supp}(g_{i,v_j})\} = \binom{\mu+2}{2},$$

which is just the number of domain points in the disk $D_\mu^T(v_j)$. We conclude that the total number of basis splines whose supports overlap T is equal to the number of domain points in T. The number of such points is $\dim \mathcal{P}_d = \binom{d+2}{2}$, and the proof is complete. $\square$

§7. The superspline space $\mathcal{S}_d^{r,\rho}(\triangle)$

In this section we show that for $d \geq 3r + 2$, an appropriate modification of the above construction leads to an LLI basis for the space of supersplines

$$\mathcal{S}_d^{r,\rho}(\triangle) := \{s \in \mathcal{S}_d^r(\triangle) : \ s \in C^{\rho_v}(v) \ \text{ for all } v \in V\},$$

with $\rho := \{\rho_v\}_{v \in V}$, where ρ_v are given integers such that $r \leq \rho_v \leq d$, and V is the set of all vertices of the triangulation $\triangle$. As in [17], we assume that

$$k_v + k_u < d \quad \text{for each pair of neighboring vertices } v, u \in V,$$

where

$$k_v := \max\{\rho_v, \mu\}, \qquad v \in V$$

with μ as in (6.1).

Next we define a determining set $\widetilde{\mathcal{M}}$ for $\mathcal{S}_d^{r,\rho}(\triangle)$ similar to the set $\mathcal{M}$ defined in Lemma 6.1. Given a triangle $T = \langle u, v, w\rangle$, let

$$\tilde{C}^T := C^T \setminus [D^T_{k_u}(u) \cup D^T_{k_v}(v) \cup D^T_{k_w}(w)].$$

Associated with u, let

$$\tilde{A}^T(u) := A^T(u) \setminus D^T_{k_u}(u),$$

with similar definitions for the other two vertices of T. Associated with the edge $e := \langle u, v\rangle$, we define

$$\begin{aligned} \tilde{G}^T_L(e) &:= G^T_L(e) \setminus D^T_{k_u}(u) \\ \tilde{G}^T_R(e) &:= G^T_R(e) \setminus D^T_{k_v}(v), \\ \tilde{E}^T(e) &:= E^T(e) \setminus [D^T_{k_u}(u) \cup D^T_{k_v}(v)], \end{aligned}$$

with similar definitions for the other edges of T.

We now define $\widetilde{\mathcal{M}}_0$ in the same way as $\mathcal{M}_0$ in Lemma 6.1, except that we use $\tilde{A}^T$ in place of A^T, etc. Let

$$\widetilde{\mathcal{M}} := \widetilde{\mathcal{M}}_0 \cup \bigcup_v D_{k_v}(v).$$

It is straightforward to check that $\widetilde{\mathcal{M}}$ is a determining set for $\mathcal{S}_d^r(\triangle)$, although it is not minimal since there are too many points in the disks $D_{k_v}(v)$.

Following our construction for $\mathcal{S}_d^r(\triangle)$, we now consider the spaces of super-splines $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$ on *cells* $\triangle_v$ associated with vertices v. Here

$$\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v) = \mathcal{P}_{\rho_v} \oplus \mathcal{V}_{r,\rho_v+1} \oplus \cdots \oplus \mathcal{V}_{r,k_v},$$

where the space $\mathcal{V}_{r,m}$ is defined in (4.1). Then using the LLI bases $\{g^{[j]}_{1,v}, \ldots, g^{[j]}_{n_j,v}\}$ for $\mathcal{V}_{r,\rho_v+j}$ constructed in Sects. 4 and 5, the same argument as in Lemma 4.1 shows that

$$\begin{aligned} \{g_{i,v}\}_{i=1}^{m_v} :=& \{x^\alpha y^\beta :\ 0 \le \alpha+\beta \le \rho_v\} \cup \{g^{[1]}_{1,v}, \ldots, g^{[1]}_{n_1,v}\} \cup \cdots \\ & \cup \{g^{[k_v-\rho_v]}_{1,v}, \ldots, g^{[k_v-\rho_v]}_{n_{k_v-\rho_v},v}\} \end{aligned}$$

is an LLI basis for $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$. Then arguing as in Theorem 6.2, it is clear that for each spline $g_{i,v} \in \mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$, there is an associated spline $B_{i,v} \in \mathcal{S}_d^{r,\rho}(\triangle)$ with $\operatorname{supp}(B_{i,v}) \subseteq \operatorname{star}(v)$ such that

$$\lambda_\eta B_{i,v} = 0, \qquad \text{all } \eta \in \widetilde{\mathcal{M}}_v := \widetilde{\mathcal{M}} \setminus D_{k_v}(v).$$

Moreover, $B_{i,v}|_T \equiv 0$ whenever $g_{i,v}|_T \equiv 0$. We now have the following analog of Theorem 6.3.

Theorem 7.1. *For each vertex $v \in \triangle$, let $\{B_{i,v}\}_{i=1}^{m_v}$ be the set of splines in $\mathcal{S}_d^{r,\rho}(\triangle)$ constructed above. In addition, for each $\xi \in \widetilde{\mathcal{M}}_0$, let B_ξ be the spline in $\mathcal{S}_d^{r,\rho}(\triangle)$ which satisfies*

$$\lambda_\eta B_\xi = \delta_{\xi,\eta}, \qquad \text{all } \eta \in \widetilde{\mathcal{M}}.$$

Then the set

$$\mathcal{B} := \{B_\xi\}_{\xi \in \widetilde{\mathcal{M}}_0} \cup \bigcup_v \{B_{i,v}\}_{i=1}^{m_v}$$

forms an LLI basis for $\mathcal{S}_d^{r,\rho}(\triangle)$.

Proof: Arguing as in Sect. 6 for $\mathcal{S}_d^r(\triangle)$, it is easy to see that $\mathcal{B}$ is a basis for $\mathcal{S}_d^{r,\rho}(\triangle)$. We now show that it is an LLI basis, by verifying (1.2). Given a triangle $T := \langle v_1, v_2, v_3 \rangle$, we first examine the number of splines B_ξ with $\xi \in \widetilde{\mathcal{M}}_0$ whose supports overlap T. By the support properties of these basis splines, it is easy to see that

$$\#\{\xi \in \widetilde{\mathcal{M}}_0 : \ T \subseteq \operatorname{supp}(B_\xi)\}$$

is equal to the sum of the cardinalities of the sets of the form $\tilde{C}^T$, $\tilde{A}^T$, $\tilde{G}_L^T$, $\tilde{G}_R^T$, and $\tilde{E}^T$. This is precisely the number of domain points in T which lie outside of the disks $D^T(v_j)$ for $j = 1, 2, 3$. Now by the local linear independence of the $\{g_{i,v_j}\}_{i=1}^{m_{v_j}}$, it follows that

$$\#\{i : \ T \subseteq \operatorname{supp}(B_{i,v_j})\} = \#\{i : \ T \subseteq \operatorname{supp}(g_{i,v_j})\} = \binom{k_v + 2}{2},$$

which is just the number of domain points in the disk $D_{k_v}^T(v_j)$. We conclude that the total number of basis splines whose supports overlap T is equal to the number of domain points in T. The number of such points is $\dim \mathcal{P}_d = \binom{d+2}{2}$, and the proof is complete. $\square$

§8. Remarks

Remark 8.1. Local linear independence was first studied for the integer shifts of a box spline, see [6,9,18].

Remark 8.2. Except for the case of supersplines with $\rho_v \geq \mu$ (for all v) treated in [17], none of the dual bases constructed in [3,16,17,20] are LLI. Indeed, the choice of determining sets in the disks $D_\mu(v)$ in those papers leads to bases for which (1.2) fails for some triangle T attached to v.

Remark 8.3. The basis for the space $S_4^1(\triangle)$ constructed in [4] is not LLI. To show this, we consider a triangulation $\triangle$ with one interior vertex v and $n \geq 5$ interior edges, where n is odd. It is easy to see that in this case the algorithm of [4] leads to a basis for $S_4^1(\triangle)$ such that each triangle $T \in \triangle$ lies in the supports of at least $n + 12 \geq 17$ basis splines. Since $\dim \mathcal{P}_4 = 15$, this basis cannot be LLI.

Remark 8.4. A basis of splines is said to be *stable* provided that the size of a spline can be bounded in terms of the size of its coefficients, and conversely. Stability is important for applications, but not so easy to achieve. The classical star-supported bases in [17] are only stable in the finite-element case $d \geq 4r+1$, $\rho_v \geq 2r$. Similarly, the LLI bases constructed here for $\mathcal{S}_d^{r,\rho}(\triangle)$ are only stable when $d \geq 4r+1$ and $\rho_v \geq 2r$ for all vertices. Stable bases for all spline and superspline spaces with $d \geq 3r+2$ have recently been constructed in [13] — see also [12] for the case $r = 1$.

Remark 8.5. It is shown in [12] that stability and local linear independence cannot generally hold simultaneously. In particular, the stable bases for certain superspline spaces constructed in [8,19] are not LLI since they fail to be least supported.

Remark 8.6. Our construction of a locally linearly independent basis can be easily adapted to the spaces of splines and supersplines on a triangulation on the sphere or a sphere-like surface introduced in [1]. Indeed, there is an isomorphism between the space $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$ and the corresponding space of homogeneous supersplines on an *orange*, such that our basis splines for $\mathcal{S}_{k_v}^{r,\rho_v}(\triangle_v)$ extend uniquely to homogeneous basis splines, see the proof of Theorem 5 in [2]. The other steps of our construction carry over in a straightforward manner, such that we get LLI bases for spaces of homogeneous splines and supersplines on arbitrary *trihedral decompositions*. These basis splines restricted to the sphere or a sphere-like surface obviously produce the desired LLI bases.

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