On almost interpolation and locally linearly independent bases

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Abstract. A characterization of almost interpolation configurations of points in terms of supports of basis functions is presented. Moreover, we show that this characterization can be significantly simplified in the case of existence of a locally linearly independent basis, so that almost interpolation sets can be constructed by taking a point in a support of each basis function. Some further results, including several equivalent definitions of a locally linearly independent system of functions, are given.

1. Introduction and main results

Let U denote a finite-dimensional space of real-valued functions defined on $K \subset \mathrm{R}^{\mathrm{d}}$. The problem of describing those configurations of points

$$T = \{t_1, \ldots, t_n\} \subset K \,, \quad n = \dim U \,,$$

(*interpolation sets* or *I-sets*), such that for any given real data $\{y_1, \ldots, y_n\}$ there exists a unique function $u \in U$ satisfying

$$u(t_i) = y_i, \; i = 1, \ldots, n,$$

has attracted considerable interest in recent years, especially for the case when $d \geq 2$ and U is a linear space of multivariate spline functions.

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In contrast to the univariate case $K \subset \mathrm{R}$, where all interpolation sets T with respect to the classical spline spaces can be characterized by the well-known Schoenberg-Whitney condition [17], it seems to be no reasonably simple way to characterize interpolation sets in the multivariate case (see [8, p. 136]). Recently, several sufficient conditions and methods to construct such configurations for multivariate spline interpolation have been developed (see [2, 7, 8, 16] and references therein).

A new approach to multivariate interpolation has been found by Sommer and Strauss [18, 19]. They introduced the concept of almost interpolation and gave a Schoenberg-Whitney type characterization of almost interpolation configurations of points for (generalized) multivariate spline spaces on polyhedral partitions, in terms of the dimensions of some restrictions of the space [19, Theorem 1.3]. Davydov [10] has extended this result to arbitrary finite-dimensional linear spaces of continuous functions on a topological space satisfying some minor restrictions. Moreover, in [10, 18] general methods of transforming a given almost interpolation set into an interpolation set are given. (See also [11].)

Given a topological space K, denote by $F(K)$ the linear space of all real functions on K. Let U be a finite-dimensional linear subspace of $F(K)$. A set

$$T = \{t_1, \ldots, t_s\} \subset K, \quad s \le \dim U,$$

is called an *almost interpolation set* (*AI-set*) *with respect to* U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, s$ there exist points $t'_i \in B_i$ such that $T' = \{t'_1, \ldots, t'_s\}$ is an interpolation set (I-set) with respect to U; *i.e.*,

$$\dim U_{|_{T'}} = s,$$

where

$$U_{|_{K'}} := \{u_{|_{K'}} : \; u \in U\}, \quad \text{for any } K' \subset K.$$

In what follows we use the notations $\operatorname{int} K'$, $\operatorname{bd} K'$ and $\overline{K'}$ for the interior, the boundary and the closure of a set $K' \subset K$, respectively. Given a function $u \in F(K)$, we set

$$\operatorname{supp} u := \overline{\{t \in K : \; u(t) \neq 0\}}.$$

Given a system of functions $u_1, \ldots, u_n \in F(K)$, we denote by

$$\operatorname{span}\{u_1, \ldots, u_n\} := \left\{ \sum_{i=1}^{n} a_i u_i : \; a_i \in \mathrm{R} \right\}$$

the linear span of this system.

One of the formulations of the univariate Schoenberg-Whitney condition says that for a system $\{B_1, \ldots, B_n\}$ of the polynomial B-splines, a set

$$T = \{t_1 < \cdots < t_n\} \subset \mathrm{R}$$

is an I-set w.r.t. span $\{B_1, \ldots, B_n\}$ if and only if

$$t_i \in \{t \in \mathrm{R} : B_i(t) \neq 0\}, \quad i = 1, \ldots, n,$$

(*support property*). It can be easily seen that AI-sets in this case are characterized by the condition

$$t_i \in \operatorname{supp} B_i, \quad i = 1, \ldots, n.$$

Sommer and Strauss [19, Proposition 1.5] gave a certain extension of this *weak support property* to multivariate splines on polyhedral partitions. Using the ideas of [10], we now present a generalization of their result to arbitrary finite-dimensional spaces of real functions on topological spaces.

Theorem 1.1 *Suppose that K is a topological space and $U \subset F(K)$ is a finite-dimensional linear space,* $\dim U = n$. *Let $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n$. Then the following conditions are equivalent.*

1) *T is an AI-set w.r.t. U.*

2) *For each basis $\{u_1, \ldots, u_n\}$ of U there exists some permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \ldots, s.$$

The main difference between Theorem 1.1 and the above-mentioned characterization of AI-sets in the case of univariate B-splines is that *each* basis of a multivariate spline space U has to be examined in order to check whether a configuration T is an AI–set. It turns out that this drawback can be overcome if U admits a locally linearly independent basis. A system of functions $\{u_1, \ldots, u_n\} \subset F(K)$ is said to be *locally linearly independent* (*LI-system*) if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B' such that $t \in B' \subset B(t)$ and the subsystem

$$\{u_i : B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent on B'. The linear span of an LI-system is called *LI-space.* (See Section 3 for the more general case of a *locally finite* system of functions as well as a discussion of various definitions of LI-systems.)

Next theorem shows that almost interpolation sets w.r.t. an arbitrary LI-space can be characterized exactly in the same way as for B-splines.

Theorem 1.2 *Let $\{u_1, \ldots, u_n\} \subset F(K)$ be a locally linearly independent system and $U = \operatorname{span}\{u_1, \ldots, u_n\}$. Let $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n$. Then the following conditions are equivalent.*

1) *T is an AI-set w.r.t. U.*

2) *$t_i \in \operatorname{supp} u_{\sigma(i)}$, $i = 1, \ldots, s$, for some permutation σ of $\{1, \ldots, n\}$*

Note that Carnicer and Peña [6] have shown that even I-sets with respect to a finite-dimensional space spanned by a locally linearly independent weak Descartes system of *univariate* continuous functions can be characterized by the support property.

It follows from Theorem 1.2 that one can construct an AI-set with respect to a given LI-system $\{u_1, \ldots, u_n\}$ by choosing a point t_i in the support of each function u_i, $i = 1, \ldots, n$. Because of this, it is important to identify those spaces of multivariate splines which admit an LI-basis. In [12] we gave several examples of such spaces, including shifts of a box spline, tensor product splines, continuous splines on simplex partitions and bivariate super splines.

The proofs of Theorem 1.1 and Theorem 1.2 are given in Section 2 and Section 3 respectively, after some necessary preparations have been made. In addition, Section 2 includes some general results on almost interpolation sets, namely, an algorithm of constructing AI-sets w.r.t. a linear subspace $V \subset U$ when an AI-set w.r.t. U is given (Lemma 2.3 and Theorem 2.6) and an algorithm of extending a given AI-set to an AI-set with more elements (Theorem 2.7). Section 3 contains, apart from the proof of Theorem 1.2, a result on the equivalence of several definitions of a locally linearly independent system of functions (Theorem 3.4), a sufficient condition for local linear independence which shows that usually it is enough to check this property only inside the cells of the partition (Theorem 3.6), and a description of the piecewise almost Chebyshev structure of any linear space of functions which admits an LI-basis (Theorem 3.9).

2. Almost interpolation sets

Assume that K is a topological space and U denotes a finite-dimensional subspace of $F(K)$, $\dim U = n$.

Definition 2.1 Let K' be any subset of K. By the *local dimension of U on K'* we mean

$$\text{l-dim}_{K'} U := \inf \{\dim U_{|_B} : K' \subset B,\ B \text{ open}\} .$$

We write $\text{l-dim}_t U$ instead of $\text{l-dim}_{\{t\}} U$. The function $\varphi : K \to \mathrm{Z}_+$ defined by $\varphi(t) := \text{l-dim}_t U$ is evidently upper semicontinuous. Moreover, it is continuous on an open everywhere dense subset $G_U \subset K$. For some further properties of local dimension see Davydov [10]. Particularly, it follows immediately from [10, Lemma 2.2] that

$$\text{(2.1)} \qquad \text{l-dim}_{K' \cup K''} U \leq \text{l-dim}_{K'} U + \text{l-dim}_{K''} U - \text{l-dim}_{K' \cap K''} U ,$$

for any $K', K'' \subset K$.

With the help of local dimension it is possible to give a “local” characterization of almost interpolation sets with respect to any finite-dimensional space U.

Theorem 2.2 [10] *Let $T = \{t_1, \ldots, t_s\} \subset K$, $s \leq n$. Then T is an AI-set w.r.t. $U \subset F(K)$ if and only if*

$$\operatorname{card} T' \leq \text{l-dim}_{T'} U$$

for any choice of a nonempty subset $T' \subset T$.

We note that Theorem 2.2 can be easily proved by using Rado theorem on independent transversals (see, e.g., [15, p. 93]).

In the proof of Theorem 1.1 we will use the following lemma.

Lemma 2.3 *Let $U \subset F(K)$ be a finite-dimensional space,* $\dim U = n$, *and let $V \subset U$ be a subspace of U,* $\dim V = n-1$. *Assume that $T = \{t_1, \ldots, t_s\} \subset$*

K is an AI-set with respect to U, $s \in \{1, \ldots, n\}$, and $\hat{T} = \{t_{i_1}, \ldots, t_{i_k}\}$ is a subset of T such that

$$\text{l-dim}_{\hat{T}}\, V = k - 1\,,$$

and

$$\text{l-dim}_{T'}\, V \geq \operatorname{card} T'$$

for any nonempty subset $T' \subset T$ with $\operatorname{card} T' < k$. Then every set $T \setminus \{t_{i_j}\}$, $j = 1, \ldots, k$, is an AI-set with respect to V.

Proof. Let T^* be an arbitrary nonempty subset of $T \setminus \{t_{i_j}\}$, $\operatorname{card} T^* = m$ for some $m \in \{1, \ldots, s-1\}$. Suppose that $\operatorname{card}(T^* \cap \hat{T}) = r$. Then $r < k$ because $t_{i_j} \notin T^*$. Therefore,

$$\text{l-dim}_{T^* \cap \hat{T}}\, V \geq r\,.$$

Since the codimension of V in U is 1 and $T^* \cup \hat{T}$ is an AI-set w.r.t. U,

$$\text{l-dim}_{T^* \cup \hat{T}}\, V \geq \text{l-dim}_{T^* \cup \hat{T}}\, U - 1 \geq \operatorname{card}(T^* \cup \hat{T}) - 1 = m + k - r - 1\,.$$

Therefore, in view of (2.1),

$$\begin{aligned} \text{l-dim}_{T^*}\, V &\geq \text{l-dim}_{T^* \cup \hat{T}}\, V + \text{l-dim}_{T^* \cap \hat{T}}\, V - \text{l-dim}_{\hat{T}}\, V \\ &\geq (m + k - r - 1) + r - (k - 1) = m = \operatorname{card} T^*\,, \end{aligned}$$

and the lemma follows from Theorem 2.2. ■

Remark 2.4 The set $\hat{T}$ in Lemma 2.3 is uniquely defined by two conditions

$$\begin{aligned} &\text{l-dim}_{\hat{T}}\, V = \operatorname{card} \hat{T} - 1\,, \\ &\text{l-dim}_{T'}\, V \geq \operatorname{card} T', \qquad \forall T' \subset \hat{T}\,. \end{aligned}$$

Indeed, if there exist two different sets $\hat{T}_1, \hat{T}_2 \subset T$ possessing this property, then by (2.1) we have

$$\begin{aligned} \text{l-dim}_{\hat{T}_1 \cup \hat{T}_2}\, V &\leq \text{l-dim}_{\hat{T}_1}\, V + \text{l-dim}_{\hat{T}_2}\, V - \text{l-dim}_{\hat{T}_1 \cap \hat{T}_2}\, V \\ &\leq (\operatorname{card} \hat{T}_1 - 1) + (\operatorname{card} \hat{T}_2 - 1) - \operatorname{card}(\hat{T}_1 \cap \hat{T}_2) \\ &= \operatorname{card}(\hat{T}_1 \cup \hat{T}_2) - 2\,, \end{aligned}$$

which contradicts the assumptions that $T \supset \hat{T}_1 \cup \hat{T}_2$ is an AI-set with respect to U, and $\dim V = \dim U - 1$.

Proof of Theorem 1.1. We first show that 2) implies 1). Assume that T fails to be an AI-set. Then, by Theorem 2.2, there exists $\tilde{T} \subset T$ such that

$$d := \text{l-dim}_{\tilde{T}}\, U < \operatorname{card} \tilde{T} =: \tilde{d}\,.$$

Let $B \supset \tilde{T}$, B open, such that $\dim U_{|_B} = d$. Since $\dim U = n$, there exist linearly independent functions $u_1, \ldots, u_{n-d}$ in U which vanish identically on B. Let us extend this system by some functions $u_{n-d+1}, \ldots, u_n \in U$ to a basis of U. By hypotheses, there exists a permutation σ of $\{1, \ldots, n\}$ such that

$$t_i \in \operatorname{supp} u_{\sigma(i)}, \quad i = 1, \ldots, s\,.$$

Since $d < \tilde{d}$, it must follow that $\sigma(i^*) \in \{1, \ldots, n-d\}$ for some $t_{i^*} \in \tilde{T}$. Hence $t_{i^*} \in \operatorname{supp}\, u_{\sigma(i^*)}$. But $u_{\sigma(i^*)} \equiv 0$ on the open set B which clearly implies that

$$\tilde{T} \subset K \setminus \operatorname{supp}\, u_{\sigma(i^*)},$$

a contradiction.

Let us show that 1) implies 2). We proceed by induction on n, assuming that the result has been proved for any finite-dimensional space of dimension at most $n-1$. Let $T = \{t_1, \ldots, t_s\}$ be an AI-set w.r.t. U and let $\{u_1, \ldots, u_n\}$ be a basis of U. Set

$$V = \operatorname{span}\{u_1, \ldots, u_{n-1}\}\,.$$

If T is an AI-set w.r.t. V (in particular, $s \le n-1$), then the result follows immediately by the induction hypothesis. Suppose that T fails to be an AI-set w.r.t. V. Then there exists a subset $\hat{T} = \{t_{i_1}, \ldots, t_{i_k}\} \subset T$ satisfying the assumptions of Lemma 2.3 for some $k \in \{1, \ldots, n\}$. We observe that $\hat{T} \cap \operatorname{supp} u_n \neq \emptyset$, because otherwise

$$\text{l-dim}_{\hat{T}}\, U = \text{l-dim}_{\hat{T}}\, V < \operatorname{card} \hat{T}\,,$$

which contradicts the supposition that T is an AI-set w.r.t. U. Thus,

$$t_{i_j} \in \operatorname{supp} u_n$$

for some $j \in \{1, \ldots, k\}$. By Lemma 2.3, $T \setminus \{t_{i_j}\}$ is an AI-set w.r.t. V, and, by the induction hypothesis, there exists a permutation σ' of $\{1, \ldots, n-1\}$ such that

$$\begin{aligned} t_i &\in \operatorname{supp} u_{\sigma'(i)}\,, & i &= 1, \ldots, i_j - 1\,, \\ t_{i+1} &\in \operatorname{supp} u_{\sigma'(i)}\,, & i &= i_j, \ldots, s-1\,. \end{aligned}$$

Then the permutation σ of $\{1,\ldots,n\}$ defined as follows,

$$\sigma(i) = \begin{cases} \sigma'(i)\,, & i = 1,\ldots,i_j - 1\,, \\ n\,, & i = i_j\,, \\ \sigma'(i-1)\,, & i = i_j + 1,\ldots,n\,, \end{cases}$$

clearly satisfies the required property. ■

Remark 2.5 We outline an alternative proof of Theorem 1.1 that relies on Hall theorem on distinct representatives. By Theorem 2.2, $T = \{t_1,\ldots,t_s\} \subset K$ is an AI-set w.r.t. U if and only if

$$\operatorname{card} T' \le \text{l-dim}_{T'} U, \qquad \forall T' \subset T\,.$$

It is not difficult to check that this last condition is equivalent to

$$\operatorname{card} T' \le \inf_{\{u_i\}_{i=1}^n} \operatorname{card}\{i :\ T' \cap \operatorname{supp} u_i \ne \emptyset\}, \qquad \forall T' \subset T,$$

where the infimum is taken over all bases $\{u_i\}_{i=1}^n$ of U. Now we fix a basis $\{u_i\}_{i=1}^n$ and set

$$A_i := \{j :\ t_i \in \operatorname{supp} u_j\}, \qquad i = 1,\ldots,s,$$

so that

$$A_i \subset \{1,2\ldots,n\}, \qquad i = 1,\ldots,s.$$

Then Theorem 1.1 follows from the Hall theorem that states the equivalence of the conditions

$$\operatorname{card}(\cup_{i\in I} A_i) \ge \operatorname{card} I, \qquad \forall I \subset \{1,2\ldots,s\},$$

and

$$\text{there exist distinct representatives } \sigma(i) \in A_i\,, \qquad i = 1,\ldots,s$$

(see, *e.g.*, [15, p. 27]).

Lemma 2.3 can be also applied to the following problem. Given an AI-set T w.r.t. $U \subset F(K)$ and a subspace $V \subset U$, find a subset $\tilde{T} \subset T$ which is an AI-set w.r.t. V.

Theorem 2.6 *Let $U \subset F(K)$ be a finite-dimensional space, and let $V \subset U$ be a subspace of U, with $\dim U - \dim V = p$. Suppose that $T = \{t_1, \dots, t_s\} \subset K$ is an AI-set with respect to U, $s > p$. Then there exists a subset $\tilde{T} \subset T$, $\operatorname{card} \tilde{T} \geq s - p$, which is an AI-set with respect to V.*

Proof. The theorem immediately follows from Lemma 2.3. ■

We now describe an algorithm of constructing AI-sets and extending a given AI-set to an AI-set with more elements.

Theorem 2.7 *Assume that $T = \{t_1, \dots, t_s\} \subset K$ $(0 \leq s < n)$ is an AI-set with respect to U. Then there exists an AI-set $\tilde{T} \subset K$ with respect to U such that $T \subset \tilde{T}$ and $\operatorname{card} \tilde{T} = n$.*

Proof. It suffices to find an AI-set $\hat{T}$ such that $T \subset \hat{T}$ and $\operatorname{card} \hat{T} = s + 1$.

If $T = \emptyset$, take any $\hat{t} \in K$ such that $u(\hat{t}) \neq 0$ for a $u \in U$, and set

$$\hat{T} := \{\hat{t}\}.$$

The statement follows immediately.

Let us consider two cases if $s \geq 1$.

Case 1. Suppose that

$$\operatorname{card} T' < \text{l-dim}_{T'} U$$

for every nonempty subset T' of T. Then taking any point $\hat{t}$ in $K \setminus T$ such that $u(\hat{t}) \neq 0$ for a $u \in U$ (which is possible since otherwise $\dim U = s < n$), we define

$$\hat{T} := T \cup \{\hat{t}\}$$

and, in view of Theorem 2.2, we have obviously obtained an AI-set with respect to U.

Case 2. Suppose that

$$\operatorname{card} T_0 = \text{l-dim}_{T_0} U \tag{2.2}$$

for some nonempty subset T_0 of T. Then there exists $\tilde{T}_0 \subset T$ which is maximal in the sense that

$$\operatorname{card} \tilde{T}_0 = \text{l-dim}_{\tilde{T}_0} U, \qquad \text{and}$$
$$T_0' \subset \tilde{T}_0 \quad \text{if} \quad T_0' \subset T \text{ and } \operatorname{card} T_0' = \text{l-dim}_{T_0'} U.$$

Indeed, if $T_0', T_0'' \subset T$ are two sets satisfying (2.2), then by (2.1)

$$\begin{aligned} \mathrm{card}\,(T_0' \cup T_0'') \leq \text{l-dim}_{T_0'\cup T_0''} U &\leq \text{l-dim}_{T_0'} U + \text{l-dim}_{T_0''} U - \text{l-dim}_{T_0'\cap T_0''} U \\ &\leq \mathrm{card}\, T_0' + \mathrm{card}\, T_0'' - \mathrm{card}\,(T_0' \cap T_0'') \\ &= \mathrm{card}\,(T_0' \cup T_0''), \end{aligned}$$

i.e., $T_0' \cup T_0''$ also satisfies (2.2).

Let $B \supset \tilde{T}_0$, B open, and

$$\mathrm{card}\, \tilde{T}_0 = \text{l-dim}_{\tilde{T}_0} U = \dim U_{|_B} .$$

Since $s < n$, we then have

$$\dim U_{|_B} \leq \mathrm{card}\, T = s < \dim U.$$

Hence there exists $\hat{u} \in U \setminus \{0\}$ such that $\hat{u} \equiv 0$ on B. Taking any point $\hat{t} \in K \setminus T$ such that $\hat{u}(\hat{t}) \neq 0$, we set

$$\hat{T} := T \cup \{\hat{t}\}.$$

By Theorem 2.2, in order to show that $\hat{T}$ is an AI-set, we have to prove the inequality

$$\mathrm{card}\, T'' \leq \text{l-dim}_{T''} U$$

for every subset T'' of $\hat{T}$. This is obviously true if $T'' = \{\hat{t}\}$ or $T'' \subset T$. Let

$$T'' = T_1 \cup \{\hat{t}\} ,$$

where $T_1 \subset T$. If $T_1 \subset \tilde{T}_0$, then

$$\mathrm{card}\,(T_1 \cup \{\hat{t}\}) = \mathrm{card}\, T_1 + 1 \leq \text{l-dim}_{T_1} U + 1 \leq \text{l-dim}_{T_1\cup\{\hat{t}\}} U ,$$

since $\hat{u}(\hat{t}) \neq 0$ and $\hat{u} \equiv 0$ on B. Otherwise, by the choice of $\tilde{T}_0$,

$$\mathrm{card}\, T_1 < \text{l-dim}_{T_1} U.$$

Therefore,

$$\mathrm{card}\,(T_1 \cup \{\hat{t}\}) = \mathrm{card}\, T_1 + 1 \leq \text{l-dim}_{T_1} U \leq \text{l-dim}_{T_1\cup\{\hat{t}\}} U,$$

which completes the proof. ∎

3. Locally linearly independent systems

Although we are mostly interested in finite systems of functions and finite-dimensional linear spaces spanned by them, the theory of local linear independence can be developed for certain infinite systems.

Let K be a topological space. We say that a system of nonzero functions $\mathcal{U} = \{u_i\}_{i\in I} \subset F(K)$, is *locally finite* if for any $t \in K$ there exists a neighborhood $B(t)$ such that the set

$$\{i \in I : B(t) \cap \text{supp } u_i \neq \emptyset\}$$

is finite. Particularly, we can consider the infinite series

$$\sum_{i\in I} a_i u_i(x), \quad x \in K,$$

taking into account the fact that for each fixed $x \in K$ only a finite number of terms is nonzero. Denote by Span $\mathcal{U}$ the linear space

$$\text{Span } \mathcal{U} := \left\{ \sum_{i\in I} a_i u_i : a_i \in \text{R} \ \text{ for all } \ \text{i} \in \text{I} \right\}.$$

It is quite clear that the local dimension

$$\text{l-dim}_{K'} U := \inf \{\dim U_{|_B} : K' \subset B,\, B \ \text{ open}\}$$

is finite for any finite $K' \subset K$ when $U = \text{Span } \mathcal{U}$. Particularly, $\varphi(t) := \text{l-dim}_t U := \text{l-dim}_{\{t\}} U$ is well-defined for such spaces.

Recall that an infinite system of functions is said to be (*algebraically*) *linearly independent* if any finite subsystem of it is linearly independent. In the theory of shift-invariant spaces another condition of linear independence has proved to be useful.

Definition 3.1 A locally finite system $\mathcal{U} = \{u_i\}_{i\in I} \subset F(K)$ is called *globally linearly independent* if

$$\sum_{i\in I} a_i u_i(x) = 0, \ x \in K, \quad \text{implies} \quad a_i = 0 \ \text{ for all } \ i \in I\,.$$

Global linear independence was first studied by de Boor and Höllig [3] for the shifts of a box spline and further investigated by other authors (see, for example, [14] and references therein). In the case of the shifts of a box spline global linear independence turned out to be equivalent to a stronger condition of *local* linear independence (de Boor and Höllig [4], Dahmen and Micchelli [9], Jia [13]). Some further references can be found in [12]. Since definitions of this notion slightly vary from paper to paper, we define it here in our own way which seems to emphasize its local nature.

Definition 3.2 A locally finite system $\mathcal{U} = \{u_i\}_{i\in I} \subset F(K)$ is said to be *locally linearly independent* (LI*-system*) if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B' such that $t \in B' \subset B(t)$ and the subsystem

$$\{u_i : B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent on B'. A linear space $U \subset F(K)$ is called LI*-space* if $U = \operatorname{Span} \mathcal{U}$ for some LI-system $\mathcal{U}$.

Ben-Artzi and Ron [1] have constructed an example of a bivariate function ϕ such that integer shifts of ϕ are globally linearly independent on R^2, but are locally linearly dependent with respect to every bounded open set $B \subset \mathrm{R}^2$.

We now present a simple example of a finite-dimensional linear space of continuous functions which admits no locally linearly independent basis and therefore fails to be an LI-space.

Example 3.3 Let $U = \operatorname{span}\{u_1, u_2\} \subset C[-1,1]$, where $u_1(t) \equiv 1$,

$$u_2(t) = \begin{cases} 0, & 0 \le t \le \frac{1}{3}, \\ t - \frac{1}{3}, & \frac{1}{3} \le t \le \frac{2}{3}, \\ \frac{1}{3}, & \frac{2}{3} \le t \le 1, \end{cases}$$
$$u_2(t) = -u_2(-t), \quad t \in [-1,0].$$

Given two nonzero functions $u_1^*, u_2^* \in U$, it is easy to check that they are linearly dependent in small neighborhoods of each of the following three points: $t_1 = -\frac{5}{6}$, $t_2 = 0$ and $t_3 = \frac{5}{6}$. On the other hand, for any nonzero $u \in U$ at least two of the points t_1, t_2, t_3 are contained in $\operatorname{supp} u$. Because of this there exists $i \in \{1,2,3\}$ such that $t_i \in \operatorname{supp} u_1^* \cap \operatorname{supp} u_2^*$ and hence u_1^* and u_2^* are not locally linearly independent.

The next theorem gives some equivalent definitions of LI-systems. We remark that Carnicer and Peña [5] have shown the equivalence of 5) and 6) below in the case of a finite system of functions $\mathcal{U}$.

Theorem 3.4 *Let $\mathcal{U} = \{u_i\}_{i\in I} \subset F(K)$ be a locally finite system of functions and let $U := \operatorname{Span} \mathcal{U}$. The following conditions are equivalent.*

1) *$\{u_i\}_{i\in I}$ is a locally linearly independent system.*

2) $\text{l-dim}_t U = \operatorname{card} \{i \in I : t \in \operatorname{supp} u_i\}$*, for any $t \in K$.*

3) $\text{l-dim}_{K'} U = \operatorname{card} I_{K'}$*, for any finite set $K' \subset K$, where*
$$I_{K'} := \{i \in I : K' \cap \operatorname{supp} u_i \neq \emptyset\}\,.$$

4) $\dim U_{|B} = \operatorname{card} I_B$*, for any open $B \subset K$ such that I_B is finite.*

5) *Given any open $B \subset K$,*
$$\sum_{i\in I} a_i u_i(x) = 0,\ x \in B, \quad \textit{implies} \quad a_i = 0 \ \textit{ for all } \ i \in I_B\,.$$

6) $\operatorname{supp}\left(\sum_{i\in I} a_i u_i\right) = \bigcup_{\substack{i\in I\\ a_i\neq 0}} \operatorname{supp} u_i$*, for any $\{a_i\}_{i\in I}$, with $a_i \in \mathrm{R}$, $i \in I$.*

Proof. The equivalence of conditions 1) and 2) immediately follows from the definitions.

Let us show that 2) implies 3). Suppose K' is a finite subset of K. Then clearly $I_{K'} = \{i \in I : K' \cap \operatorname{supp} u_i \neq \emptyset\}$ is a finite set. Consider any open set $B \supset K'$ such that
$$\dim U_{|B} = \text{l-dim}_{K'} U\,,$$
$$B \cap \operatorname{supp} u_i = \emptyset \quad \text{for any } i \notin I_{K'}\,.$$

Since obviously
$$\dim U_{|B} \leq \operatorname{card} I_{K'}\,,$$
it will be sufficient to check the opposite inequality
$$\dim U_{|B} \geq \operatorname{card} I_{K'}\,.$$

To this end we show that the functions $u_{i|_B}$, $i \in I_{K'}$, are linearly independent. Suppose that for some real a_i, $i \in I_{K'}$,

$$\sum_{i \in I_{K'}} a_i u_i(x) \equiv 0\,, \quad x \in B\,.$$

For each $t \in K'$, let

$$I(t) := \{i \in I :\ t \in \operatorname{supp} u_i\}\,,$$

and let $B(t)$ be an open neighborhood of t such that

$$B(t) \subset B\,,$$
$$\dim U_{|_{B(t)}} = \text{l-dim}_t\, U\,,$$
$$B(t) \cap \operatorname{supp} u_i = \emptyset \quad \text{for any } i \notin I(t)\,.$$

It follows from 2) that

$$\dim U_{|_{B(t)}} = \text{l-dim}_t\, U = \operatorname{card} I(t)\,.$$

Hence, the functions $u_{i|_{B(t)}}$, $i \in I(t)$, are linearly independent. Because of this, $a_i = 0$ for all $i \in I(t)$. Since

$$\bigcup_{t \in K'} I(t) = I_{K'}\,,$$

we have $a_i = 0$ for all $i \in I_{K'}$, which confirms the linear independence of $u_{i|_B}$, $i \in I_{K'}$, and completes the proof of 3).

We now show that 3) implies 4). Let $B \subset K$ be any open set such that $I_B = \{i \in I : B \cap \operatorname{supp} u_i \neq \emptyset\}$ is finite. It is clear that $\dim U_{|_B} \leq \operatorname{card} I_B$. Assuming 3) to hold, we prove the opposite inequality. To this end we take a point $t_i \in B \cap \operatorname{supp} u_i$ for each $i \in I_B$ and set $K' := \{t_i\}_{i \in I_B}$. Then

$$\dim U_{|_B} \geq \text{l-dim}_{K'}\, U = \operatorname{card} I_{K'} = \operatorname{card} I_B\,.$$

Let us show that 4) implies 6). Suppose that 4) holds. Let

$$t \in \operatorname{supp}(\sum_{i \in I} a_i u_i)\,.$$

Since the system $\{u_i\}_{i\in I}$ is locally finite, there exists an open neighborhood B of t such that I_B is finite. Therefore,

$$t \in \operatorname{supp}(\sum_{i\in I_B} a_i u_i) \subset \bigcup_{\substack{i\in I_B \\ a_i\neq 0}} \operatorname{supp} u_i \,,$$

so that

$$\operatorname{supp}(\sum_{i\in I} a_i u_i) \subset \bigcup_{\substack{i\in I \\ a_i\neq 0}} \operatorname{supp} u_i \,.$$

Let now

$$t \notin \operatorname{supp}(\sum_{i\in I} a_i u_i) \,.$$

Then there exists an open neighborhood B of t such that I_B is finite and

$$\sum_{i\in I} a_i u_i(x) = \sum_{i\in I_B} a_i u_i(x) = 0 \quad \text{for any } \ x \in B \ .$$

By 4) the functions

$$u_i|_B \,, \quad i \in I_B \,,$$

are linearly independent. Therefore, $a_i = 0$ for all $i \in I_B$. If $t \in \operatorname{supp} u_i$, then $i \in I_B$ and $a_i = 0$, so that

$$t \notin \bigcup_{\substack{i\in I \\ a_i\neq 0}} \operatorname{supp} u_i \,,$$

which proves 6).

Next, we show that 6) implies 5). Assume that 6) is satisfied. Let an open $B \subset K$ be given. If

$$\sum_{i\in I} a_i u_i(x) = 0 \quad \text{for any } \ x \in B \ ,$$

then, by 6),

$$B \subset \left(K \setminus \operatorname{supp} \sum_{i\in I} a_i u_i \right) = \bigcap_{\substack{i\in I \\ a_i\neq 0}} (K \setminus \operatorname{supp} u_i) \ .$$

Thus, $B \cap \operatorname{supp} u_i = \emptyset$ for any $i \in I$ such that $a_i \neq 0$, which proves 5).

Since 1) evidently follows from 5), the proof is complete. ■

We are now in position to give a very short proof of the second of our two main theorems, Theorem 1.2.

Proof of Theorem 1.2. It is clear by Theorem 1.1 that 1) implies 2). Consider now any subset $\tilde{T} = \{t_{i_1}, \ldots, t_{i_r}\}$ of T. Assume that 2) holds. Then

$$t_{i_j} \in \operatorname{supp} u_{\sigma(i_j)}, \quad j = 1, \ldots, r,$$

for some permutation σ of $\{1, \ldots, n\}$. Since by hypothesis $\{u_1, \ldots, u_n\}$ is a locally linearly independent basis of U, we have by Theorem 3.4, 3), that

$$\text{l-dim}_{\tilde{T}}\, U \geq r = \operatorname{card} \tilde{T}\,.$$

In view of Theorem 2.2 this implies that T is an AI-set. ■

Remark 3.5 It is possible to prove the implication 2) $\Rightarrow$ 1) of Theorem 1.2 without using Theorem 2.2. We proceed by induction on s. For $s = 1$ the implication is evidently true. Let $\{u_i\}_{i=1}^n$ be an LI-system, and let $t_i \in \operatorname{supp} u_i$, $i = 1, \ldots, n$. Suppose that $T_{s-1} := \{t_i\}_{i=1}^{s-1}$ is an AI-set w.r.t. $U_{s-1} := \{u_i\}_{i=1}^{s-1}$, while $T_s := \{t_i\}_{i=1}^{s}$ w.r.t. $U_s := \{u_i\}_{i=1}^{s}$ is not. Then there exist neighborhoods $\{B_i\}_{i=1}^s$ of t_i's, and a set $T'_{s-1} := \{t'_i\}_{i=1}^{s-1}$, with $t'_i \in B_i$, such that

$$a_s := \det\{u_i(t'_j)\}_{i,j=1}^{s-1} \neq 0,$$

and

$$b(t'_s) := \det\{u_i(t'_j)\}_{i,j=1}^{s} = 0, \qquad \forall t'_s \in B_s.$$

Setting $t'_s = t$ and expanding the determinant $b(t)$ with respect to the last column, we obtain

$$b(t) = \sum_{i \leq s} c_i u_i(t) = \sum_{i \leq s:\, B_s \cap \operatorname{supp} u_i \neq \emptyset} c_i u_i(t) \equiv 0, \qquad t \in B_s,$$

where

$$B_s \cap \operatorname{supp} u_s \neq \emptyset, \qquad c_s = a_s \neq 0,$$

so that the system U_s is not linearly independent on B_s, which contradicts Theorem 3.4, 5).

Turning back to the comparison of various definitions of local linear independence, we note that the definitions 1) – 5) of Theorem 3.4 are in fact listed in the order of increasing proportion of open sets B such that the subsystem

$$\mathcal{U}_B := \{u_i : B \cap \operatorname{supp} u_i \neq \emptyset\}$$

has to be tested for linear independence on B. Thus, the weakest in this sense is definition 1) or, equivalently, 2), where only small neighborhoods of every point $t \in K$ have to be considered. We now want to make one step further and show that some undesirable points $t \in K$ can often be taken out of consideration.

We say that a locally finite system $\mathcal{U} = \{u_i\}_{i \in I} \subset F(K)$ is *linearly independent in the neighborhood of* $t \in K$ if for any neighborhood $B(t)$ of t there exists an open set B' such that $t \in B' \subset B(t)$, and $\mathcal{U}_B$ is linearly independent on B'. This is obviously equivalent to the condition

$$\text{l-dim}_t U = \operatorname{card}\{i \in I : t \in \operatorname{supp} u_i\}.$$

The following theorem shows that usually it is enough to check linear independence only in the neighborhoods of points $t \in G_U$, where G_U denotes the set of all points of continuity of local dimension $\varphi(t) = \text{l-dim}_t U$. This is useful in applications to spline spaces, in particular, since local dimension is much easier to count inside the cells of a partition (*i.e.*, inside the components of G_U) than on its edges (see, *e.g.*, the proof of Theorem 3.11 in [12]). We also note that the original definition of local linear independence by de Boor and Höllig [4] requires linear independence of $\mathcal{U}_B$ only for open B contained in some cell of the partition.

Theorem 3.6 *Let $\mathcal{U} = \{u_i\}_{i \in I} \subset F(K)$ be a locally finite system of functions, $U = \operatorname{Span} \mathcal{U}$. Assume that*

$$\overline{\operatorname{int}(\operatorname{supp} u_i)} = \operatorname{supp} u_i, \quad i \in I. \tag{3.1}$$

If $\mathcal{U}$ is linearly independent in the neighborhood of every $t \in G_U$, then $\mathcal{U}$ is an LI-system.

Proof. We first note that G_U is an open and everywhere dense subset of K by [10, Proposition 4.2]. (In fact, this is the only property of G_U we will use in the proof.) Let us consider the locally finite system

$$\mathcal{U}' := \{u_{i|_{G_U}}\}_{i \in I} \subset F(G_U).$$

Since G_U is open,

$$\operatorname{supp} u_i|_{G_U} = G_U \cap \operatorname{supp} u_i, \qquad i \in I.$$

Suppose now that $\mathcal{U}$ fails to be locally linearly independent. Then there exists an open set $B \subset K$ such that $\mathcal{U}_B$ is linearly dependent on B. Since the sets $\operatorname{int}(\operatorname{supp} u_i)$ and G_U are open and everywhere dense in $\operatorname{supp} u_i$ and K, respectively, we have

$$\begin{aligned}
\{i : B \cap \operatorname{supp} u_i \neq \emptyset\} &= \{i : B \cap \operatorname{int}(\operatorname{supp} u_i) \neq \emptyset\} \\
&= \{i : B \cap G_U \cap \operatorname{int}(\operatorname{supp} u_i) \neq \emptyset\} \\
&= \{i : B \cap G_U \cap \operatorname{supp} u_i \neq \emptyset\} \\
&= \{i : B' \cap \operatorname{supp} u_i|_{G_U} \neq \emptyset\},
\end{aligned}$$

where $B' := B \cap G_U$. Therefore, the subsystem of $\mathcal{U}'$,

$$\mathcal{U}'_{B'} = \{u_i|_{G_U} : B' \cap \operatorname{supp} u_i|_{G_U} \neq \emptyset\} = \{u_i|_{G_U} : B \cap \operatorname{supp} u_i \neq \emptyset\},$$

is linearly dependent on B', which implies, by Theorem 3.4, that $\mathcal{U}'$ is not an LI-system, *i.e.*, $\mathcal{U}'$ (and, therefore, $\mathcal{U}$) fails to be linearly independent in the neighborhood of some point $t \in G_U$, contrary to the assumptions. ■

We note that condition (3.1) holds for any system of *continuous* functions u_i, since in that case the set $\{t \in K : u_i(t) \neq 0\}$ is open and everywhere dense in $\operatorname{supp} u_i$.

In the rest of this section we will study the sectional structure of LI-spaces. For some general results on the piecewise almost Chebyshev structure of finite-dimensional linear spaces of real functions see [10, 11].

Suppose $\mathcal{U} = \{u_i\}_{i \in I} \subset F(K)$ is a locally finite (not necessarily an LI-) system, $U = \operatorname{Span} \mathcal{U}$. For any subset $I' \subset I$, let

$$G_{I'} := \left(\bigcap_{i \in I'} \operatorname{int}(\operatorname{supp} u_i) \right) \bigcap \left(\bigcap_{i \in I \setminus I'} K \setminus \operatorname{supp} u_i \right).$$

In the case $I' = \emptyset$ we obviously have

$$G_\emptyset = K \setminus \bigcup_{i \in I} \operatorname{supp} u_i .$$

(It may happen that $G_\emptyset \neq \emptyset$.)

Lemma 3.7 *$G_{I'}$ is an open subset of K. Moreover, if I' is infinite, then $G_{I'} = \emptyset$.*

Proof. Since $\{u_i\}_{i\in I}$ is a locally finite system, we have

$$\bigcap_{i\in I'} \operatorname{int}(\operatorname{supp} u_i) = \emptyset \quad \text{if } I' \text{ is infinite}.$$

Therefore, $G_{I'}$ can be nonempty only for a finite I'. Suppose now that I' is finite. Let $t \in G_{I'}$ and let B be an open neighborhood of t such that $I_B := \{i \in I : B \cap \operatorname{supp} u_i \neq \emptyset\}$ is finite. Then

$$B \subset \bigcap_{i\in I\setminus I_B} K \setminus \operatorname{supp} u_i \,.$$

Therefore,

$$B' := B \bigcap \left(\bigcap_{i\in I'} \operatorname{int}(\operatorname{supp} u_i) \right) \bigcap \left(\bigcap_{i\in I_B\setminus I'} K \setminus \operatorname{supp} u_i \right)$$

is also an open neighborhood of t. Moreover, since $(I\setminus I_B) \cup (I_B \setminus I') \supset (I\setminus I')$, we have $B' \subset G_{I'}$. ■

If $G_{I'} \neq \emptyset$, then evidently

$$(3.2) \qquad G_{I'} \cap \operatorname{supp} u_i \neq \emptyset \quad \text{if and only if} \quad i \in I'.$$

Therefore, $U_{|G_{I'}}$ is a finite-dimensional space if I' is finite and $G_{I'} \neq \emptyset$.

Definition 3.8 A finite-dimensional linear space $U \subset F(K)$, $\dim U = n$, is said to be an *almost Haar* (or *almost Chebyshev*) *space* if every set $T = \{t_1, \ldots, t_n\} \subset K$ is an almost interpolation set w.r.t. U.

Theorem 3.9 *Let $\mathcal{U} = \{u_i\}_{i\in I} \subset F(K)$ be a locally linearly independent system, $U = \operatorname{Span} \mathcal{U}$, and let G_U denote the set of all points of continuity of $\varphi(t) = \text{l-dim}_t U$. Then*

$$(3.3) \qquad \begin{aligned} G_U &= \bigcup_{\substack{I'\subset I \\ I' \text{ finite}}} G_{I'}, \\ K \setminus G_U &= \bigcup_{i\in I} \operatorname{bd}(\operatorname{supp} u_i). \end{aligned}$$

Moreover, for any finite $I' \subset I$, *if* $G_{I'} \neq \emptyset$, *then* $U_{|_{G_{I'}}}$ *is an almost Haar space of dimension* $\operatorname{card} I'$, *and*

$$\text{(3.4)} \qquad \text{l-dim}_t\, U = \operatorname{card} I' \,, \quad t \in G_{I'} \,.$$

Proof. Let us note that (3.4) is a consequence of Theorem 3.4, 2), since

$$\{i \in I : t \in \operatorname{supp} u_i\} = I' \,, \quad t \in G_{I'} \,.$$

Moreover, by Theorem 3.4, 4) and (3.2), we have

$$\dim U_{|_{G_{I'}}} = \operatorname{card} I'.$$

If $T = \{t_1, \ldots, t_s\} \subset G_{I'}$, $s = \operatorname{card} I'$, then $T \subset \operatorname{supp}\, u_i$ for all $i \in I'$. Therefore, in view of Theorem 1.2, T is an almost interpolation set w.r.t. $U_{|_{G_{I'}}}$, which shows that $U_{|_{G_{I'}}}$ is an almost Haar space of dimension $\operatorname{card} I'$.

In order to prove the first part of the theorem, we first show that

$$K \setminus \bigcup_{I' \subset I} G_{I'} = \bigcup_{i \in I} \operatorname{bd}(\operatorname{supp} u_i) \,.$$

Indeed, by the distributive law,

$$\begin{aligned} \bigcup_{i \in I} \operatorname{bd}(\operatorname{supp} u_i) &= \bigcup_{i \in I} \{\operatorname{supp} u_i \cap (K \setminus \operatorname{int}(\operatorname{supp} u_i))\} \\ &= \bigcap_{I' \subset I} \left\{ \left(\bigcup_{i \in I'} \operatorname{supp} u_i \right) \bigcup \left(\bigcup_{i \in I \setminus I'} K \setminus \operatorname{int}(\operatorname{supp} u_i) \right) \right\} \\ &= K \setminus \bigcup_{I' \subset I} G_{I'} \,. \end{aligned}$$

It remains to prove that $\text{l-dim}_t\, U$ is continuous on

$$\bigcup_{\substack{I' \subset I \\ I' \text{ finite}}} G_{I'}$$

and discontinuous on

$$\bigcup_{i \in I} \operatorname{bd}(\operatorname{supp} u_i) \,.$$

The former follows immediately from (3.4). In order to prove the latter, suppose that $t \in \bigcup_{i\in I} \mathrm{bd}\,(\mathrm{supp}\, u_i)$. Let $I = I_1 \cup I_2 \cup I_3$ such that

$$\begin{aligned} t &\in \mathrm{bd}\,(\mathrm{supp}\, u_i)\,, & i \in I_1\,, \\ t &\in \mathrm{int}\,(\mathrm{supp}\, u_i)\,, & i \in I_2\,, \\ t &\notin \mathrm{supp}\, u_i\,, & i \in I_3\,. \end{aligned}$$

Then, by Theorem 3.4,

$$\text{l-dim}_t\, U = \mathrm{card}\, I_1 + \mathrm{card}\, I_2\,.$$

There exists a neighborhood B of t such that for any $x \in B$,

$$\begin{aligned} x &\in \mathrm{int}\,(\mathrm{supp}\, u_i)\,, & i \in I_2\,, \\ x &\notin \mathrm{supp}\, u_i\,, & i \in I_3\,. \end{aligned}$$

On the other hand, for any open $B' \subset B$ there exists a point $x' \in B'$ such that

$$x' \notin \mathrm{supp}\, u_{i'}\,, \quad \text{for some} \quad i' \in I_1\,.$$

Then

$$\text{l-dim}_{x'}\, U \le \mathrm{card}\, I_1 - 1 + \mathrm{card}\, I_2 = \text{l-dim}_t\, U - 1\,,$$

from which it follows that t is a point of discontinuity of local dimension. ■

Remark 3.10 It is interesting to compare (3.3) with the decomposition of G_U into the union of its connected components G_s,

$$G_U = \bigcup_{s\in S} G_s, \tag{3.5}$$

in [10, Eq. (4.9)] when U is a finite-dimensional LI-space. Both decompositions obviously coincide if every $G_{I'}$ is connected. Otherwise, each $G_{I'}$ has to be further decomposed into its connected components in order to get (3.5) from (3.3). However, sometimes (3.3) can do the job of revealing the spline-like structure of U much better than (3.5). Let, for example, $\mathcal{U}$ be the set of cardinal B-splines restricted to a rational interval $[a, b] \cap \mathbb{Q}$. Then (3.3) produces usual partition of $[a, b] \cap \mathbb{Q}$ into knot intervals while (3.5) is not very useful in this case since all G_s are singletons. We also mention that (3.3) is necessarily a finite decomposition as soon as U is a finite-dimensional LI-space.

We finally remark that some other properties of (finite) locally linearly independent systems, specifically *minimal* (or *"least"*) *supportedness*, have been studied by Carnicer and Peña [5].

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