Interpolation by Bivariate Linear Splines

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Dedicated to Professor L. L. Schumaker on the occasion of his 60th Birthday

Abstract. We give a characterization of Lagrange interpolation sets for the spaces of continuous bivariate linear splines on regular triangulations. The characterization is based on a complete description of the zero sets of such splines.

§1. Introduction

Let K be a bounded simply connected polygonal domain in $\mathbb{R}^2$, and let $\Delta = \{K_i\}_{i\in I}$ be a regular triangulation of K, *i.e.*,

$$K = \bigcup_{i\in I} K_i,$$

where I is a finite set, each K_i is a closed triangle, and no vertex of K_i lies in the interior of K_j or in the interior of an edge of K_j, for all $i, j \in I$. Let us note that K does not have holes (since K has a connected complement in $\mathbb{R}^2$), and Δ is strongly connected, *i.e.*, for any two triangles K_i, K_j, $i, j \in I$ there exists a sequence of triangles $K_{i_1}, \dots, K_{i_l}$ such that $i = i_1$, $j = i_l$, $\{i_1, \dots, i_l\} \subset I$, and K_{i_σ} and $K_{i_{\sigma+1}}$ have a common edge for all $\sigma \in \{1, \dots, l-1\}$.

We are interested in Lagrange interpolation by the elements of the space $S(\Delta)$ of all continuous real-valued linear spline functions with respect to Δ. Thus,

$$S(\Delta) = \{s \in C(K) : s_{|K_i} \in \pi_1,\ i \in I\},$$

where π_1 denotes the space of bivariate linear polynomials. We say that a set $T = \{t_1, \dots, t_n\} \subset K$, where $n = \dim S(\Delta)$, is an interpolation set (I-set) w.r.t. $S(\Delta)$ if for any given data $y_1, \dots, y_n \in \mathbb{R}$ there exists a unique function $s \in S(\Delta)$ such that

$$s(t_i) = y_i, \qquad i = 1, \dots, n.$$

While in the univariate case interpolation sets with respect to a spline space can be characterized by the well-known Schoenberg-Whitney condition, the multivariate situation requires further investigation (see, *e.g.*, [1,4]). As shown in [1, p. 136], even in the simplest case of the bivariate linear spline space $S(\Delta)$ no simple extension of the Schoenberg-Whitney theorem is possible. It is well-known that interpolation from $S(\Delta)$ is always possible at the vertices $v_1, \ldots, v_n$ of the triangulation Δ, *i.e.*, $V = \{v_1, \ldots, v_n\}$ is an I-set w.r.t. $S(\Delta)$. However, the problem of determining *all* I-sets for $S(\Delta)$ does not seem to admit a simple solution. An algorithm for constructing rather general interpolation sets w.r.t. $S(\Delta)$ was proposed in [2]. A general approach to constructing I-sets, that applies in particular to $S(\Delta)$, consists in first determining an almost interpolation set and then transforming it into an I-set (see [3,4,7]).

Some simple conditions on the location of interpolation and almost interpolation points can be given in terms of the supports of the Courant hat functions (see Section 2 below).

On the other hand, the problem of determining I-sets w.r.t. $S(\Delta)$ is essentially equivalent to the problem of describing zero sets

$$Z(s) := \{t \in K : s(t) = 0\}$$

of functions $s \in S(\Delta)$. Indeed, it follows from basic linear algebra that $T = \{t_1, \ldots, t_n\} \subset K$ is an I-set w.r.t. $S(\Delta)$ if and only if there exists no spline $s \in S(\Delta) \setminus \{0\}$ such that $T \subset Z(s)$.

In the main part of the paper (Section 3) we give a necessary and sufficient condition for a set $Z \subset K$ to be a zero set $Z(s)$ of an appropriate spline $s \in S(\Delta)$ (Theorem 3.5). This immediately implies a characterization of all I-sets w.r.t. $S(\Delta)$ (Theorem 3.6).

§2. Interpolation Sets and Supports of Courant Hat Functions

Let $V = \{v_1, \ldots, v_n\}$ denote the set of all vertices of the triangulation Δ. It is well-known that $\dim S(\Delta) = n$ and

$$S(\Delta) = \operatorname{span}\{u_1, \ldots, u_n\}$$

where the u_i's are the Courant hat functions defined uniquely by

$$u_i(v_j) = \delta_{ij}, \qquad i, j = 1, \ldots, n.$$

The support of u_i, $\operatorname{supp} u_i := \overline{\{t \in K : u_i(t) \neq 0\}}$, is easily seen to be the star of the vertex v_i, *i.e.*,

$$\operatorname{supp} u_i = \operatorname{star}(v_i) := \bigcup \{K_j : \ v_i \in K_j\}.$$

Moreover,

$$\{t \in K : u_i(t) \neq 0\} = \operatorname{int}_K \operatorname{star}(v_i), \tag{2.1}$$

where $\operatorname{int}_K M$ denotes the interior of M with respect to K.

A necessary condition for I-sets can be easily derived from the following general result.

Theorem 2.1. *Let U be a linear space of real functions defined on a set K, and let $u_1, \ldots, u_n$ be a basis for U. If $T = \{t_1, \ldots, t_n\} \subset K$ is an I-set w.r.t. U, then there exists a permutation σ of $\{1, \ldots, n\}$ such that*

$$u_{\sigma(i)}(t_i) \neq 0, \qquad i = 1, \ldots, n.$$

Proof: The theorem is obviously true for $n = 1$. We proceed by induction on n and assume that the statement has been proved for $n - 1$. Let

$$u(t) := D(u_1, \ldots, u_n; t_1, \ldots, t_{n-1}, t), \qquad t \in K,$$

where we use the notation

$$D(u_1, \ldots, u_n; z_1, \ldots, z_n) := \det(u_i(z_j))_{i,j=1}^n.$$

Obviously,

$$u(t) = c_1 u_1(t) + \cdots + c_n u_n(t) \in U,$$

with

$$c_i := (-1)^{n+i} D(u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n; t_1, \ldots, t_{n-1}).$$

Since $\{t_1, \ldots, t_n\}$ is an I-set, it follows that $u(t_n) \neq 0$. Hence $u_\ell(t_n) \neq 0$ for some ℓ such that $c_\ell \neq 0$. Then $\{t_1, \ldots, t_{n-1}\}$ is an I-set w.r.t.

$$\tilde{U} := \operatorname{span}\{u_1, \ldots, u_{l-1}, u_{l+1}, \ldots, u_n\}.$$

Induction hypotheses on these basis functions and the points $\{t_1, \ldots, t_{n-1}\}$ give the desired permutation. □

In view of (2.1) we immediately get

Corollary 2.2. *If $T = \{t_1, \ldots, t_n\} \subset K$ is an I-set w.r.t. $S(\Delta)$, then there exists a permutation σ of $\{1, \ldots, n\}$ such that*

$$t_i \in \operatorname{int}_K \operatorname{star}(v_{\sigma(i)}), \qquad i = 1, \ldots, n.$$

The converse of this statement is not true in general (see Example 9.2 in [1, p. 137] as well as Example 3.6 below), which shows that the simple analogue of the Schoenberg-Whitney condition is not valid. However, the situation is much nicer if we switch to almost interpolation.

Definition 2.3. A set $T = \{t_1, \ldots, t_n\} \subset K$ is called an almost interpolation set w.r.t. $S(\Delta)$ if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, n$, there exist points $t_i' \in B_i$ such that $T' = \{t_1', \ldots, t_n'\}$ is an I-set w.r.t. $S(\Delta)$.

The next result follows immediately from Theorem 1.2 in [6] in view of the obvious fact that $\{u_1, \ldots, u_n\}$ is a locally linearly independent basis for $S(\Delta)$, *i.e.*, for every triangle K_j the functions $\{u_i : \ K_j \subset \operatorname{supp} u_i\}$ are linearly independent on K_j.

Theorem 2.4. *Any set $T = \{t_1, \dots, t_n\} \subset K$ is an almost interpolation set w.r.t. $S(\Delta)$ if and only if there exists a permutation σ of $\{1, \dots, n\}$ such that*

$$t_i \in \text{star}(v_{\sigma(i)}), \qquad i = 1, \dots, n.$$

Thus, by taking $t_i \in \text{int}_K \operatorname{supp} u_i = \text{int}_K \text{star}(v_i)$, $i = 1, \dots, n$, we "almost always" get an I-set $T = \{t_1, \dots, t_n\}$. The simplest example of such an I-set is provided by the choice $t_i = v_i$, $i = 1, \dots, n$. In fact, as shown in [7], T is guaranteed to be an I-set if t_i's lie in the supports of u_i's not farther than "half-way" from v_i. (We give here this result with its short proof for the sake of completeness.)

Theorem 2.5. [7] *If*

$$t_i \in L_i := \{t \in K : u_i(t) > \frac{1}{2}\}, \qquad i = 1, \dots, n,$$

then $T = \{t_1, \dots, t_n\}$ is an I-set w.r.t. $S(\Delta)$.

Proof: It can be easily seen that $\sum_{j=1}^{n} u_j(t) = 1$ for all $t \in K$. Since $0 \leq u_i(t) \leq 1$ for all $i = 1, \dots, n$ and all $t \in K$, it follows that

$$0 \leq \sum_{\substack{j=1 \\ j \neq i}}^{n} u_j(t_i) = 1 - u_i(t_i) < \frac{1}{2}.$$

Therefore, the matrix $M := (u_j(t_i))_{i,j=1}^{n}$ is diagonally dominant. Thus, M is a regular matrix which implies that T is an I-set. □

§ 3. Interpolation Sets and Zero Sets

As mentioned in the introduction, $T = \{t_1, \dots, t_n\} \subset K$ is an I-set w.r.t. $S(\Delta)$ if and only if there does not exist a nontrivial $s \in S(\Delta)$ such that $s(t_i) = 0$ for all $i = 1, \dots, n$, *i.e.*, $T \subset Z(s)$. In order to characterize I-sets it is therefore a fundamental task to study the zero sets $Z(s)$ of the splines $s \in S(\Delta)$. In this section we will completely describe these sets and then be able to give a characterization of all I-sets w.r.t. $S(\Delta)$.

Suppose that $s \in S(\Delta)$. For every $i \in I$ we denote by $s_i \in \pi_1$ the linear polynomial (defined on $\mathbb{R}^2$) that coincides with s on K_i, *i.e.*, $s_{i|K_i} = s_{|K_i}$. Let $l_i(s) \subset \mathbb{R}^2$ be the zero set of s_i,

$$l_i(s) := Z(s_i), \qquad i \in I.$$

Obviously, $l_i(s)$ is either the empty set (if s_i is a nonzero constant), or a straight line (if s_i is nonconstant), or the set $\mathbb{R}^2$ (if s_i is the zero function). Let us denote the set of all such objects by $\mathcal{Z}$, *i.e.*,

$$\mathcal{Z} := \{Z(p) : \ p \in \pi_1\}.$$

Thus, to every $s \in S(\Delta)$ there corresponds a family of subsets of $\mathbb{R}^2$,

$$\mathcal{L}(s) := \{l_i(s) : \; i \in I\} \subset \mathcal{Z}.$$

The zero set of s is then given by

$$Z(s) = \bigcup_{i \in I} \lambda_i(s),$$

where

$$\lambda_i(s) := l_i(s) \cap K_i, \qquad i \in I.$$

We are now interested in the following

Question. *Let an arbitrary family*

$$\mathcal{L} := \{l_i\}_{i \in I} \subset \mathcal{Z}$$

be given. What are the necessary and sufficient conditions on $\mathcal{L}$ that ensure the existence of a spline $s \in S(\Delta)$ such that $\mathcal{L} = \mathcal{L}(s)$?

First, it follows immediately from the continuity of $s \in S(\Delta)$ that $\mathcal{L} = \mathcal{L}(s)$ necessarily satisfies the following two conditions.

Condition A. For any $i, j \in I$, if the triangles K_i and K_j have a common edge e, then

$$l_i \cap l(e) = l_j \cap l(e),$$

where $l(e)$ denotes the straight line containing e.

Condition B. For any $i, j \in I$, if the triangles K_i and K_j have a common vertex v, then

$$l_i \cap \{v\} = l_j \cap \{v\}.$$

Indeed, if, for example, $K_i \cap K_j = e$ and $l_i(s) \cap l(e) = \{z\}$, then necessarily $l_j(s) \cap l(e) = \{z\}$ since s_i and s_j coincide on $l(e)$. See also Fig. 3.1.

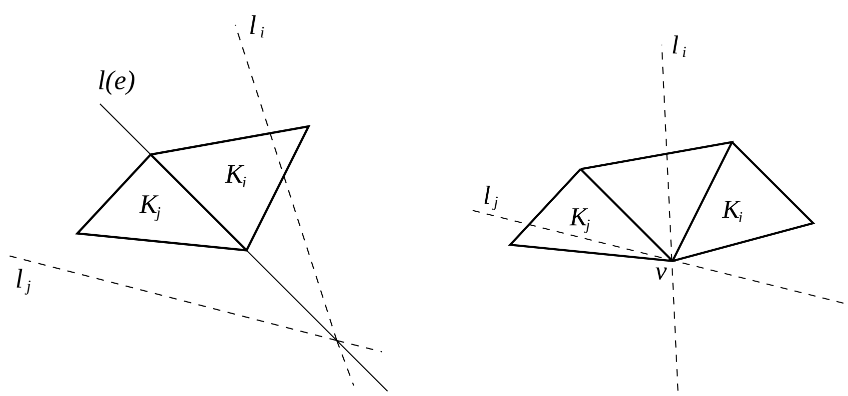

Fig. 3.1. Conditions A and B.

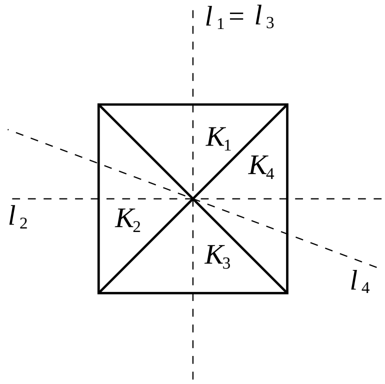

Fig. 3.2. A and B are not sufficient.

Unfortunately, Conditions A and B are not sufficient for $\mathcal{L}$ to admit a spline $s \in S(\Delta)$ with $\mathcal{L} = \mathcal{L}(s)$. For example, it is not difficult to see that for the triangulation $\Delta = \{K_1, K_2, K_3, K_4\}$ and set $\mathcal{L} = \{l_1, l_2, l_3, l_4\} \subset \mathcal{Z}$ shown in Fig. 3.2 there exists no spline $s \in S(\Delta)$ such that $\mathcal{L} = \mathcal{L}(s)$ despite the fact that Conditions A and B are satisfied. (In fact $l_i(s) = l_i$, $i = 1, 2, 3$, implies $l_4(s) = l_2 \neq l_4$.)

Thus, an additional condition on $\mathcal{L}$ has to be imposed. In order to formulate it in full generality, we need some preparation.

Definition 3.1. Let M be a nonempty connected subset of K. The shell of M, denoted by shell(M), is the intersection of all simply connected subsets of K containing the set

$$K_M := \bigcup_{\substack{i \in I \\ K_i \cap M \neq \emptyset}} K_i.$$

It is easy to see that K_M is necessarily connected and shell(M) is just the union of K_M and all its holes. Therefore, the boundary of shell(M) is a closed polygonal line without loops. For any vertex v_i of Δ, $\text{shell}(\{v_i\}) = \text{star}(v_i)$.

We now associate with $\mathcal{L}$ a set $L^* \subset K$ defined by

$$L^* := \bigcup_{i \in I} \lambda_i^*,$$

where λ_i^* denotes the convex hull of $l_i \cap K_i \cap \{v_1, \ldots, v_n\}$. Thus, each λ_i^* is either K_i if $l_i = \mathbb{R}^2$, or an edge e of K_i if $l_i \cap K_i = e$, or $\{v_j\}$, where v_j is a vertex of K_i, if $l_i \cap K_i = \{v_j\}$, or $\emptyset$ if no vertex of K_i lies on l_i. We note that if $\mathcal{L} = \mathcal{L}(s)$ for some $s \in S(\Delta)$, then $L^* \subset Z(s)$.

Definition 3.2. Let $L^1, \ldots, L^r$ be connected components of L^*, such that $L^* = L^1 \cup \cdots \cup L^r$. For every $j = 1, \ldots, r$, the boundary of shell(L^j) is called a cycle w.r.t. $\mathcal{L}$ if $L^j \subset \text{int}\,\text{shell}(L^j)$.

It is easily seen that the boundary of shell(L^j) is a cycle if and only if $L^j \subset \text{int}\,K$. See Fig. 3.3 for a typical zero set of a spline $s \in S(\Delta)$ along with the corresponding set L^* and the (only) cycle w.r.t. $\mathcal{L}(s)$.

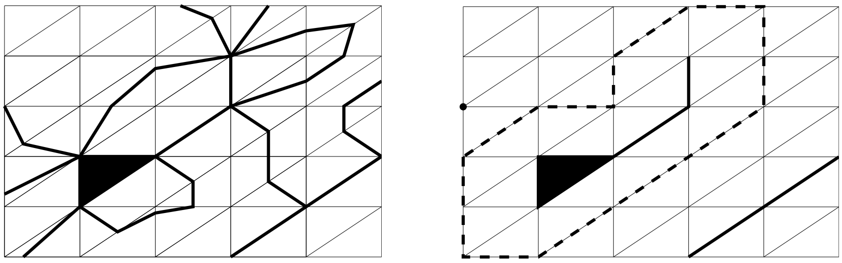

Fig. 3.3. Zero set (to the left), L^* and a cycle (to the right).

We are now ready to determine the third condition that necessarily holds for $\mathcal{L}$ if $\mathcal{L} = \mathcal{L}(s)$, with some $s \in S(\Delta)$.

Let σ be a cycle w.r.t. $\mathcal{L}$. Then σ is the boundary of shell(L^j) for some connected component L^j of L^*, and, hence, σ is a closed polygonal line without loops. Let us denote its vertices by $v_1, \ldots, v_m$ in counterclockwise order. Set $v_{m+1} := v_1$. By the definition of a cycle, $v_i \notin L^j$, $i = 1, \ldots, m$, but for every i there exists a triangle $K_{\mu_i} \in \Delta$ such that $K_{\mu_i} = \text{con}(v_i, v_{i+1}, w_i)$ (the convex hull of vertices v_i, v_{i+1}, w_i) and $w_i \in L^j$. It follows that l_{μ_i} contains w_i, and $l_{\mu_i} \cap \{v_i, v_{i+1}\} = \emptyset$. (If, for example, $v_i \in l_{\mu_i}$, then the edge e with vertices v_i and w_i would be a subset of l_{μ_i}, and hence $e \subset \lambda^*_{\mu_i}$, which would imply $e \subset L^j$.) Particularly, l_{μ_i} is a straight line. Furthermore, denote by $l(v_i, v_{i+1})$ the straight line that contains v_i and v_{i+1}. Let

$$\hat{I} := \{i \in \{1, \ldots, m\} : \; l_{\mu_i} \text{ and } l(v_i, v_{i+1}) \text{ are not parallel}\}.$$

For all $i \in \hat{I}$ we denote by z_i the intersection point of l_{μ_i} and $l(v_i, v_{i+1})$ (see Fig. 3.4). By the above arguments, $z_i \notin \{v_i, v_{i+1}\}$, $i = 1, \ldots, m$.

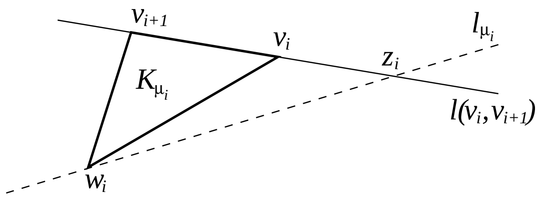

Fig. 3.4. Intersection of l_{μ_i} and $l(v_i, v_{i+1})$.

To simplify our arguments, we make the following definition.

Definition 3.3. A cycle σ w.r.t. $\mathcal{L}$ is called singular if

$$\prod_{i \in \hat{I}} \varepsilon_i \frac{\rho(v_{i+1}, z_i)}{\rho(v_i, z_i)} = 1, \tag{3.1}$$

where

$$\varepsilon_i := \begin{cases} -1, & \text{if } z_i \in \text{con}(v_i, v_{i+1}), \\ \;\;1, & \text{otherwise ,} \end{cases}$$

and $\rho(z, w)$ denotes the usual Euclidian distance between two points in the plane.

Our third condition reads as follows.

Condition C. Every cycle w.r.t. $\mathcal{L}$ is singular.

If now $\mathcal{L} = \mathcal{L}(s)$ for some $s \in S(\Delta)$, then $s(v_i) \neq 0$, $i = 1,\dots,m$. Consider, as before, the linear polynomials s_{μ_i} that coincide with s on K_{μ_i}, $i = 1,\dots,m$. If $i \in \hat{I}$, then s_{μ_i} restricted to $l(v_i, v_{i+1})$ is a linear function vanishing at z_i. If, otherwise, $i \notin \hat{I}$, then the zero line l_{μ_i} of s_{μ_i} is parallel to $l(v_i, v_{i+1})$, and, hence, s_{μ_i} is constant on $l(v_i, v_{i+1})$. Therefore,

$$\frac{s(v_{i+1})}{s(v_i)} = \frac{s_{\mu_i}(v_{i+1})}{s_{\mu_i}(v_i)} = \begin{cases} \varepsilon_i \frac{\rho(v_{i+1}, z_i)}{\rho(v_i, z_i)}, & \text{if } i \in \hat{I}, \\ 1, & \text{otherwise.} \end{cases}$$

Since $v_{m+1} = v_1$, we have

$$1 = \prod_{i=1}^{m} \frac{s(v_{i+1})}{s(v_i)} = \prod_{i \in \hat{I}} \varepsilon_i \frac{\rho(v_{i+1}, z_i)}{\rho(v_i, z_i)},$$

and Condition C holds.

We are now in position to answer the above Question.

Theorem 3.4. *Given $\mathcal{L} = \{l_i\}_{i \in I} \subset \mathcal{Z}$, there exists a spline $s \in S(\Delta)$ such that $\mathcal{L} = \mathcal{L}(s)$ if and only if Conditions A, B and C hold. Moreover, if Conditions A, B and C are satisfied, then the dimension of the subspace of splines $s \in S(\Delta)$ such that $l_i \subset Z(s_i)$, $i \in I$, is equal to the number of connected components of $K \setminus L^*$.*

The necessity of Conditions A, B and C has been shown above. Sufficiency as well as the second statement of the theorem will be proved in the next section.

As immediate consequences of the first statement of Theorem 3.4 we now give a characterization of zero sets of splines $s \in S(\Delta)$ and a characterization of I-sets w.r.t. $S(\Delta)$.

Theorem 3.5. *A set $Z \subset K$ is a zero set of a spline $s \in S(\Delta)$ if and only if there exists $\mathcal{L} = \{l_i\}_{i \in I} \subset \mathcal{Z}$ such that Conditions A, B and C are satisfied and $Z \cap K_i = l_i \cap K_i$ for all $i \in I$.*

Proof: If $s \in S(\Delta)$, then $Z(s) = \bigcup_{i \in I}(l_i \cap K_i)$, with $\{l_i\}_{i \in I} = \mathcal{L}(s)$. Conversely, if $\mathcal{L} = \{l_i\}_{i \in I} \subset \mathcal{Z}$ satisfies Conditions A, B and C, then by Theorem 3.4 there exists $s \in S(\Delta)$ such that $\mathcal{L} = \mathcal{L}(s)$. If now $Z \cap K_i = l_i \cap K_i$, $i \in I$, then $Z = \bigcup_{i \in I}(l_i \cap K_i) = Z(s)$. $\square$

Theorem 3.6. *Let $T = \{t_1, \dots, t_n\} \subset K$. Then T is an I-set w.r.t. $S(\Delta)$ if and only if there does not exist $\mathcal{L} = \{l_i\}_{i \in I} \subset \mathcal{Z}$ such that*

1) *$l_i \neq \mathbb{R}^2$ for at least one $i \in I$,*
2) *$T \cap K_i \subset l_i$ for all $i \in I$, and*
3) *$\mathcal{L}$ satisfies Conditions A, B and C.*

Proof: The result follows from Theorem 3.5 since T is an I-set w.r.t. $S(\Delta)$ if and only if there exists no spline $s \in S(\Delta) \setminus \{0\}$ such that $T \subset Z(s)$. $\square$

To illustrate Theorems 3.5 and 3.6, we consider the following examples.

Example 3.7. Let Δ consist of two triangles K_1, K_2 with a common edge e. Then $\dim S(\Delta) = 4$. According to Theorem 3.5, $Z \subset K := K_1 \cup K_2$ is a zero set of a spline $s \in S(\Delta)$ if and only if $Z = (l_1 \cap K_1) \cup (l_2 \cap K_2)$, where $\mathcal{L} = \{l_1, l_2\} \subset \mathcal{Z}$ satisfies Conditions A, B and C. It follows that there are the following possibilities:

1) $l_1 = l_2 = \mathbb{R}^2$, $Z = K$, $s \equiv 0$,
2) $l_1 = \mathbb{R}^2$, $l_2 = l(e)$, $Z = K_1$, or $l_2 = \mathbb{R}^2$, $l_1 = l(e)$, $Z = K_2$,
3) both l_1 and l_2 are straight lines that either intersect at a point on $l(e)$, or are parallel to each other and to $l(e)$, and, consequently, Z is either an interval or the union of two intervals or the empty set,
4) $l_1 = \emptyset$, l_2 is a straight line parallel to $l(e)$, $Z = l_2 \cap K_2$, or $l_2 = \emptyset$, l_1 is a straight line parallel to $l(e)$, $Z = l_1 \cap K_1$,
5) $l_1 = l_2 = \emptyset$, $Z = \emptyset$, s is a nonzero constant.

We now describe the I-sets $T = \{t_1, t_2, t_3, t_4\} \subset K$ w.r.t. $S(\Delta)$. By Corollary 2.2 not more than three points t_i lie in the same triangle K_1 or K_2. Therefore, we distinguish two cases.

a) Suppose that exactly three points t_i lie in the same triangle, *i.e.*, say, $t_1, t_2, t_3 \in K_1$ and $t_4 \in \operatorname{int}_K K_2$. Then it is easy to see by Theorem 3.6 and the above description of zero sets that T is an I-set if and only if the points t_1, t_2, t_3 are not collinear.
b) Suppose that $t_1, t_2 \in \operatorname{int}_K K_1$ and $t_3, t_4 \in \operatorname{int}_K K_2$. Then T is an I-set if and only if $l(t_1, t_2) \cap l(e) \neq l(t_3, t_4) \cap l(e)$. (See Fig. 3.5.)

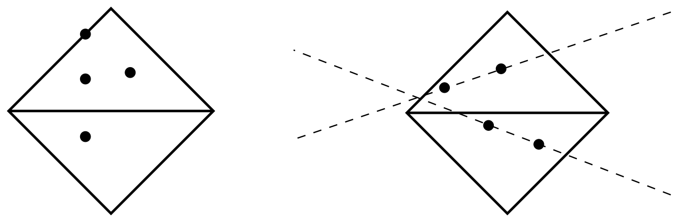

Fig. 3.5. Typical I-sets in Example 3.7.

We note that Condition C does not apply to Example 3.7.

Example 3.8. Let K be the square $[-1, 1]^2$, and let Δ be defined by drawing in the two diagonals. Then $\dim S(\Delta) = 5$. Consider

$$T = \left\{(0,0), (\tfrac{1}{2}, 0), (-\tfrac{1}{2}, 0), (0, \tfrac{1}{2}), (0, -\tfrac{1}{2})\right\},$$

see Fig. 3.6. It is easy to see that conditions 1)–3) of Theorem 3.6 are satisfied for $\mathcal{L} = \{l_1, l_2, l_3, l_4\}$, where $l_1 = l_3 = \{(0, y) : \; y \in \mathbb{R}\}$ and $l_2 = l_4 = \{(x, 0) : x \in \mathbb{R}\}$. Particularly, Condition C holds for $\mathcal{L}$ since $L^* = \{(0,0)\}$, the only cycle w.r.t. $\mathcal{L}$ is formed by the boundary of K, with vertices $v_1 = (1,1)$, $v_2 = (-1,1)$, $v_3 = (-1,-1)$ and $v_4 = (1,-1)$, and since the intersection points $z_1 = (0,1)$, $z_2 = (-1,0)$, $z_3 = (0,-1)$ and $z_4 = (1,0)$ satisfy (3.1). Therefore, by Theorem 3.6, T is not an I-set w.r.t. $S(\Delta)$. However, if we move a point in

T from the corresponding line l_i, then we get an I-set. Consider, for example, the set

$$T' = \{(0,0), (\tfrac{1}{2}, \varepsilon), (-\tfrac{1}{2}, 0), (0, \tfrac{1}{2}), (0, -\tfrac{1}{2})\}$$

with some $0 < |\varepsilon| < \frac{1}{2}$. Then the only $\mathcal{L}'$ satisfying 1) and 2) of Theorem 3.6 as well as Conditions A and B is $\mathcal{L}' = \{l_1, l_2, l_3, l_4'\}$, where l_4' is the line $y = 2\varepsilon x$. Since $\mathcal{L}'$ does not satisfy Condition C, it follows that T' is an I-set.

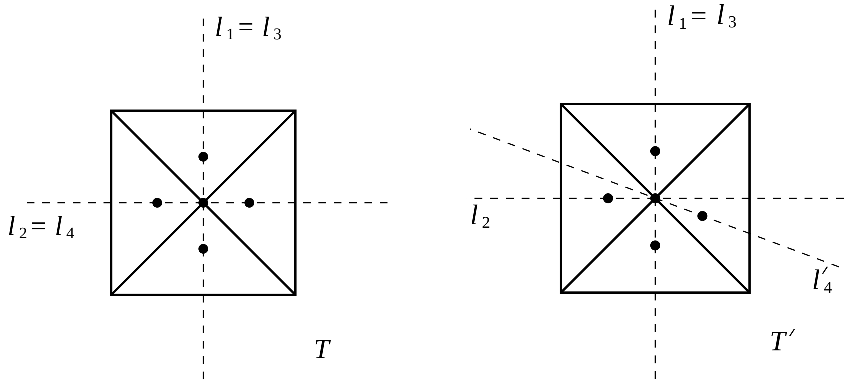

Fig. 3.6. Non-I-set T and I-set T'.

§ 4. Proof of Theorem 3.4

The necessity of Conditions A, B and C has been shown in Section 3. In order to prove sufficiency, let us assume that $\mathcal{L} = \{l_i\}_{i \in I} \subset \mathcal{Z}$ satisfies Conditions A, B and C. We now construct a function $s \in S(\Delta)$ such that $\mathcal{L} = \mathcal{L}(s)$, *i.e.*,

$$l_i = Z(s_i) \quad \text{for all} \quad i \in I. \tag{4.1}$$

The proof is rather lengthy and we divide it into several parts.

(I) Corresponding to the family $\mathcal{L}$ let us consider $L^* \subset K$ defined in Section 3. If $L^* = K$, then $l_i = \mathbb{R}^2$ for all $i \in I$, and the spline $s \equiv 0$ has the desired properties. Therefore, we assume that $K \setminus L^* \neq \emptyset$. Let

$$K \setminus L^* = K^{[1]} \cup \ldots \cup K^{[q]},$$

where $K^{[j]}$ are the connected components of $K \setminus L^*$, $j = 1, \ldots, q$. Then

$$K^{[j]} = \left(\bigcup_{i \in I^{[j]}} K_i \right) \setminus L^*$$

for some subsets $I^{[j]}$ of I such that $l_i \neq \mathbb{R}^2$ for all $i \in I^{[j]}$.

For every $j \in \{1, \ldots, q\}$ we will construct a function $s^{[j]} \in S(\Delta)$ satisfying

$$\begin{aligned} s^{[j]}{}_{|K_i} &\equiv 0 \quad \text{for all} \quad i \notin I^{[j]}, \\ Z(s_i^{[j]}) &= l_i \quad \text{for all} \quad i \in I^{[j]}, \end{aligned} \tag{4.2}$$

where $s_i^{[j]}$ denotes a linear polynomial such that $s_i^{[j]}{}_{|K_i} \equiv s^{[j]}{}_{|K_i}$. Then every spline

$$s = \sum_{j=1}^{q} a_j s^{[j]}, \quad \text{with} \quad a_j \neq 0, \quad j = 1, \ldots, q,$$

satisfies (4.1).

To describe the construction of s it is therefore only necessary to consider a single set $K^{[j]}$. For simplicity of notation, we write $\tilde{K}$, $\tilde{I}$, $\tilde{s}$ and $\tilde{s}_i$ instead of $K^{[j]}$, $I^{[j]}$, $s^{[j]}$ and $s_i^{[j]}$, respectively.

(II) We want to construct $\tilde{s}$ step by step, passing from one triangle to another. To this end the triangles K_i, $i \in \tilde{I}$, have to be appropriately ordered. The order that we need is described in the following lemma. (Recall that $H \subset \mathbb{R}^2$ is called a hole of a connected set $M \subset \mathbb{R}^2$ if H is a bounded connected component of $\mathbb{R}^2 \setminus M$.)

Lemma 4.1. *There exists a sequence of subsets*

$$\tilde{K}^\gamma := \left(\bigcup_{i \in I_\gamma} K_i \right) \setminus L^* \subset \tilde{K}, \qquad \gamma = 1, \ldots, p,$$

where $I_1 \subset \ldots \subset I_p = \tilde{I}$ *and* $\operatorname{card} I_\gamma = \gamma$, *such that for all* $\gamma = 1, \ldots, p$,

1) $\tilde{K}^\gamma$ *is strongly connected, and*
2) *every hole* H *of* $\operatorname{clos} \tilde{K}^\gamma$ *(the closure of* $\tilde{K}^\gamma$*) is also a hole of* $\operatorname{clos} \tilde{K}$.

Proof: We set $I_1 = \{i_1\}$ for some $i_1 \in \tilde{I}$. The set $\tilde{K}^1 = K_{i_1} \setminus L^*$ obviously satisfies 1) and 2).

We proceed by induction and assume that for some $\gamma \leq p-1$ the set $\tilde{K}^\gamma$ satisfying 1) and 2) has been constructed. We now have to determine some $i_{\gamma+1} \in \tilde{I} \setminus I_\gamma$ such that

$$\tilde{K}^{\gamma+1} := \tilde{K}^\gamma \cup (K_{i_{\gamma+1}} \setminus L^*)$$

also satisfies 1) and 2).

Observe first that there exists some $\nu \in \tilde{I} \setminus I_\gamma$ such that $K_\nu \setminus L^*$ has a common edge with $\tilde{K}^\gamma$ since otherwise $\tilde{K}^\gamma$ would be a connected component of $K \setminus L^*$ contradicting the assumption that $\gamma \leq p-1$. If now $\tilde{K}^\gamma \cup (K_\nu \setminus L^*)$ satisfies 2), then we set $i_{\gamma+1} := \nu$ and are finished.

Suppose that 2) fails. Then there exists a hole H of $\operatorname{clos}(\tilde{K}^\gamma \cup K_\nu)$ which is not a hole of $\operatorname{clos} \tilde{K}$, *i.e.*, there is at least one triangle K_i, $i \in \tilde{I}$, lying in $\operatorname{clos} H$. Moreover, by induction hypothesis, H is not a hole of $\operatorname{clos} \tilde{K}^\gamma$. (In particular, K_ν has a common edge with $\operatorname{clos} H$.) Since $\tilde{K}$ is strongly connected, it follows that there exists $\nu_1 \in \tilde{I} \setminus I_\gamma$ such that $K_{\nu_1} \subset \operatorname{clos} H$ and K_{ν_1} has a common edge with $\tilde{K}^\gamma$.

If $\tilde{K}^\gamma \cup (K_{\nu_1} \setminus L^*)$ satisfies 2), then we set $i_{\gamma+1} := \nu_1$ and are finished. Otherwise, there must exist a hole H_1 of $\operatorname{clos}(\tilde{K}^\gamma \cup K_{\nu_1})$ which is not a hole of $\tilde{K}$, and

$$H_1 \subset H, \quad H_1 \neq H.$$

Then we find $\nu_2 \in \tilde{I} \setminus I_\gamma$ such that $K_{\nu_2} \subset \operatorname{clos} H_1$ and has a common edge with $\tilde{K}^\gamma$.

By a repeated application of these arguments we obtain a sequence of indices ν_i and holes H_i satisfying

$$H_{i+1} \subset H_i, \quad H_{i+1} \neq H_i.$$

Since the number of triangles in Δ is finite, this process must stop after a finite number of steps, which guarantees the existence of some $\nu_r \in \tilde{I} \setminus I_\gamma$ such that $\tilde{K}^{\gamma+1} := \tilde{K}^\gamma \cup (K_{\nu_r} \setminus L^*)$ satisfies 1) and 2), and the proof is complete. □

(III) We are now ready to proceed to the construction of $\tilde{s}_i$, $i \in I$. Obviously,

$$\tilde{s}_i \equiv 0 \quad \text{for all} \quad i \notin \tilde{I}.$$

Thus, we have to determine $\tilde{s}_i$ for $i \in \tilde{I}$. Recall that

$$l_i \neq \mathbb{R}^2 \quad \text{for all} \quad i \in \tilde{I}.$$

Using the sequences $(\tilde{K}^\gamma)_{\gamma=1}^p$ and $(I_\gamma)_{\gamma=1}^p$ constructed in Lemma 4.1, we assume without loss of generality that

$$I_\gamma = \{1, \ldots, \gamma\}, \qquad \gamma = 1, \ldots, p.$$

We first determine $\tilde{s}_1$. If $l_1 = \emptyset$, then we set

$$\tilde{s}_1(t) \equiv 1.$$

If l_1 is a straight line, then we choose a point $\bar{t} \in K_1 \setminus l_1$ and define $\tilde{s}_1$ as a unique linear function such that

$$\tilde{s}_1(\bar{t}) = 1, \qquad \tilde{s}_1(t) = 0 \quad \text{for all} \quad t \in l_1.$$

Assume that for some $\gamma \leq p-1$ the linear polynomials $\tilde{s}_1, \ldots, \tilde{s}_\gamma$ have been appropriately chosen such that

$$Z(\tilde{s}_i) = l_i \quad \text{for all} \quad i = 1, \ldots, \gamma,$$

and the piecewise linear function

$$\tilde{s}^\gamma(t) := \tilde{s}_i(t), \quad \text{for} \quad t \in K_i, \qquad i = 1, \ldots, \gamma,$$

is continuous on $\operatorname{clos}\tilde{K}^\gamma = \bigcup_{i=1}^{\gamma} K_i$. It is easy to see that

$$\tilde{s}^\gamma(t) = 0 \quad \text{for all} \quad t \in L^* \cap \operatorname{clos}\tilde{K}^\gamma.$$

We now want to determine $\tilde{s}_{\gamma+1} \in \pi_1$ such that $Z(\tilde{s}_{\gamma+1}) = l_{\gamma+1}$ and $\tilde{s}_{\gamma+1}$ continuously extends $\tilde{s}^\gamma$ to $K_{\gamma+1}$. We will see that such a linear polynomial always exists and is furthermore unique. Since $\tilde{K}^{\gamma+1} = \tilde{K}^\gamma \cup (K_{\gamma+1} \setminus L^*)$ is strongly connected (by Lemma 4.1), $\tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^*$ have at least one common edge. (Here we do not count the edges that lie in L^*.) Moreover, $\tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^*$ cannot have three edges in common, for otherwise $\operatorname{int} K_{\gamma+1}$ would be a hole of $\operatorname{clos}\tilde{K}^\gamma$ and consequently, by Lemma 4.1, a hole of $\operatorname{clos}\tilde{K}$, in contradiction with the fact that $\operatorname{int} K_{\gamma+1} \subset \tilde{K}^{\gamma+1} \subset \tilde{K}$. Thus, $\tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^*$ always have either one or two common edges. We distinguish these two cases.

Case 1: $\tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^*$ have exactly one common edge e.

Since $e \not\subset L^*$, we have $e \setminus l_{\gamma+1} \neq \emptyset$. Let us choose a point $t_0 \in e \setminus l_{\gamma+1}$. We define $\tilde{s}_{\gamma+1}$ to be a unique linear polynomial such that

$$Z(\tilde{s}_{\gamma+1}) = l_{\gamma+1} \quad \text{and} \quad \tilde{s}_{\gamma+1}(t_0) = \tilde{s}^\gamma(t_0).$$

We have to show that the extended piecewise linear function

$$\tilde{s}^{\gamma+1}(t) := \begin{cases} \tilde{s}^\gamma(t), & \text{if } t \in \operatorname{clos}\tilde{K}^\gamma, \\ \tilde{s}_{\gamma+1}(t), & \text{if } t \in K_{\gamma+1} \end{cases} \tag{4.3}$$

is continuous on $\operatorname{clos}\tilde{K}^{\gamma+1} = \operatorname{clos}\tilde{K}^\gamma \cup K_{\gamma+1}$. Since $\tilde{s}^\gamma$ is continuous on $\operatorname{clos}\tilde{K}^\gamma$ by induction hypotheses, we only have to ensure the continuity of $\tilde{s}^{\gamma+1}$ across the edges and vertices of $K_{\gamma+1}$ that lie in $\operatorname{clos}\tilde{K}^\gamma$. Let $K_{\gamma+1} = \operatorname{con}(v_1, v_2, v_3)$ and $e = \operatorname{con}(v_1, v_2)$. The continuity of $\tilde{s}^{\gamma+1}$ across the edge e easily follows from Condition A. It may happen that one of the other two edges of $K_{\gamma+1}$, say $e' := \operatorname{con}(v_1, v_3)$, lies in $\operatorname{clos}\tilde{K}^\gamma$. Then necessarily $e' \subset L^*$, and hence, by Condition A, $e' \subset l_{\gamma+1}$. Therefore, $\tilde{s}^\gamma(t) = \tilde{s}_{\gamma+1}(t) = 0$ for all $t \in e'$, which guarantees the continuity across e'. It remains to prove that $\tilde{s}^{\gamma+1}$ is continuous at v_3 if $v_3 \in \operatorname{clos}\tilde{K}^\gamma$. This follows from the above arguments if at least one of the edges $e' = \operatorname{con}(v_1, v_3)$ or $e'' = \operatorname{con}(v_2, v_3)$ lies in $\operatorname{clos}\tilde{K}^\gamma$. If, otherwise, none of e' and e'' lies in $\operatorname{clos}\tilde{K}^\gamma$, then $\operatorname{clos}\tilde{K}^{\gamma+1}$ must have a hole H whose boundary includes one of these edges, say e'. (See Fig. 4.1.) By Lemma 4.1 H is also a hole of $\operatorname{clos}\tilde{K}$. Since $\tilde{K}$ is a connected component of $K \setminus L^*$, and K is simply connected, it follows that the boundary of H lies in L^*. In particular, $v_3 \in L^*$. Condition B then implies that $v_3 \in l_{\gamma+1}$ and $\tilde{s}^\gamma(v_3) = \tilde{s}_{\gamma+1}(v_3) = 0$, so that $\tilde{s}^{\gamma+1}$ is continuous at v_3.

Case 2: $\tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^*$ have two common edges e_1 and e_2.

Let $e_i = \operatorname{con}(v, v_i)$, $i = 1, 2$. Assume first that $v \notin L^*$. Then we define $\tilde{s}_{\gamma+1}$ as a unique linear polynomial such that

$$Z(\tilde{s}_{\gamma+1}) = l_{\gamma+1} \quad \text{and} \quad \tilde{s}_{\gamma+1}(v) = \tilde{s}^\gamma(v).$$

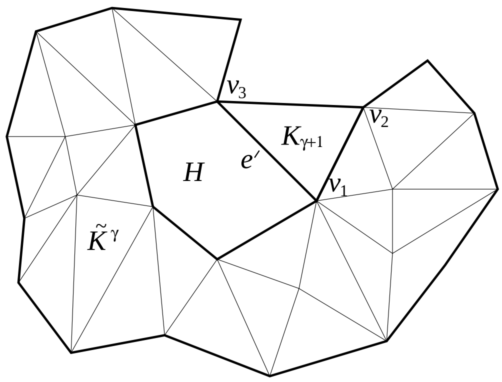

Fig. 4.1. Hole H in Case 1.

The continuity of $\tilde{s}^{\gamma+1}$ given by (4.3) on clos $\tilde{K}^{\gamma+1}$ can be checked in the same way as in Case 1.

Assume now that $v \in L^*$. Since $e_1, e_2 \not\subset L^*$, we have

$$K_{\gamma+1} \cap L^* = \{v\}.$$

Define $\tilde{s}_{\gamma+1}$ as a unique linear polynomial such that

$$\tilde{s}_{\gamma+1}(v) = 0, \qquad \tilde{s}_{\gamma+1}(v_i) = \tilde{s}^{\gamma}(v_i), \quad i = 1, 2. \tag{4.4}$$

Since $\tilde{s}^{\gamma}(v) = 0$, it is obvious that the piecewise polynomial function $\tilde{s}^{\gamma+1}$ given by (4.3) is continuous on clos $\tilde{K}^{\gamma+1}$. Thus, it remains to show that

$$Z(\tilde{s}_{\gamma+1}) = l_{\gamma+1}. \tag{4.5}$$

To this end we denote by $\hat{L}$ the connected component of L^* that contains v. Let, furthermore, $\mathrm{sm}(\hat{L})$ be the union of $\hat{L}$ and all its holes. (Thus, $\mathrm{sm}(\hat{L})$ is the intersection of all simply connected subsets of K containing $\hat{L}$.)

Lemma 4.2. $\mathrm{sm}(\hat{L})$ *is a hole of* $\tilde{K}^{\gamma+1}$. *Moreover,*

$$\mathrm{shell}(\hat{L}) \setminus \mathrm{sm}(\hat{L}) \subset \tilde{K}^{\gamma+1}. \tag{4.6}$$

Proof: We first note that (4.6) follows immediately from the first statement of the lemma in view of the fact that the boundary of $\mathrm{shell}(\hat{L})$ cannot contain points of L^*.

Denote by K_p and K_q the triangles in $\tilde{K}^{\gamma}$ that are attached to e_1 and e_2 respectively. Since $\tilde{K}^{\gamma}$ is strongly connected, there exists a sequence of triangles

$$K_p = K_{i_1}, K_{i_2}, \ldots, K_{i_r} = K_q, \quad \text{with} \quad i_j \in I_\gamma, \quad j = 1, \ldots, r,$$

such that K_{i_j} and $K_{i_{j+1}}$ have a common edge lying in $\tilde{K}^{\gamma}$ for all $j = 1, \ldots, r-1$. We set

$$\hat{K} := \bigcup_{j=1}^{r} K_{i_j}.$$

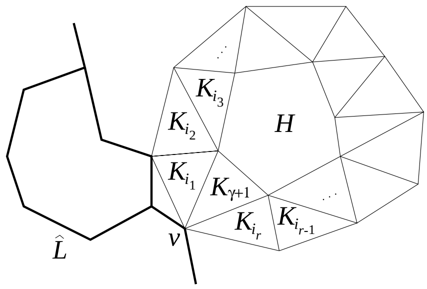

Fig. 4.2. $\operatorname{int} K_{\gamma+1}$ cannot lie in a hole of $\hat{K}$.

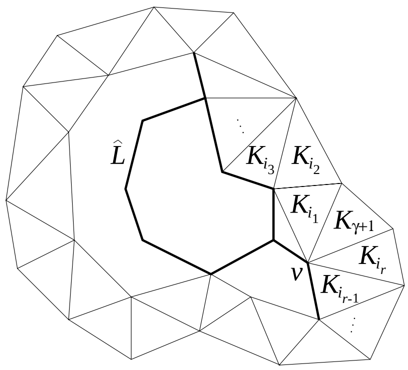

Fig. 4.3. $\hat{L}$ must lie in a hole of $(\hat{K} \cup K_{\gamma+1}) \setminus L^*$.

Assume for a moment that $\operatorname{int} K_{\gamma+1}$ lies in a hole H of $\hat{K}$ (see Fig. 4.2). Since $\hat{K} \setminus L^* \subset \tilde{K}^\gamma$ and $K_{\gamma+1} \setminus L^* \subset \tilde{K} \setminus \tilde{K}^\gamma$, we can then find a hole H' of $\operatorname{clos} \tilde{K}^\gamma$, such that $\operatorname{int} K_{\gamma+1} \subset H' \subset H$. Then H' is not a hole of $\operatorname{clos} \tilde{K}$, contrary to Lemma 4.1.

Therefore, $\operatorname{int} K_{\gamma+1}$ cannot lie in a hole of $\hat{K}$. Consequently, $K_{\gamma+1} \setminus L^*$ cannot lie in a hole of $\hat{L}$. Moreover, it follows that $\hat{L}$ must lie in a hole $\hat{H}$ of $(\hat{K} \cup K_{\gamma+1}) \setminus L^*$ (see Fig. 4.3). Then $\operatorname{sm}(\hat{L}) \subset \hat{H}$. Since

$$(\hat{K} \cup K_{\gamma+1}) \setminus L^* \subset \tilde{K}^{\gamma+1} \quad \text{and} \quad \operatorname{sm}(\hat{L}) \cap \tilde{K}^{\gamma+1} = \emptyset,$$

there exists a hole $\hat{H}'$ of $\tilde{K}^{\gamma+1}$ such that $\operatorname{sm}(\hat{L}) \subset \hat{H}' \subset \hat{H}$. The proof will be completed if we show that

$$\operatorname{sm}(\hat{L}) = \hat{H}'.$$

Recall that $\tilde{K}^{\gamma+1}$ has the form $U \setminus L^*$, where U is the union of certain closed triangles in Δ. If $\operatorname{int} \hat{H}' = \emptyset$, then $\hat{H}'$ is easily seen to be a connected component of L^*. Since $\hat{L} \subset \hat{H}'$ is a connected component of L^*, it follows that $\operatorname{sm}(\hat{L}) = \hat{L} = \hat{H}'$ as desired. Now suppose that $\operatorname{int} \hat{H}' \neq \emptyset$. Then $\hat{H}' \setminus \operatorname{clos}(\operatorname{int} \hat{H}') \subset L^*$. Moreover, $\operatorname{int} \hat{H}'$ is a hole of $\operatorname{clos} \tilde{K}^{\gamma+1}$. Hence, by Lemma 4.1, $\operatorname{int} \hat{H}'$ is also a hole of $\operatorname{clos} \tilde{K}$. Since $\tilde{K}$ is a connected component of $K \setminus L^*$, we have $\operatorname{clos}(\operatorname{int} \hat{H}') \subset \operatorname{sm}(\hat{L})$. The desired conclusion now follows from the fact that $\hat{H}'$ is connected. $\square$

We now continue the proof of (4.5). It follows from Lemma 4.2 that the boundary of $\operatorname{shell}(\hat{L})$ is a cycle σ w.r.t. $\mathcal{L}$. Then σ is a closed polygonal line without loops. Let $v_1, \ldots, v_m$ denote the vertices of σ, with v_1 and v_2 being the vertices of $K_{\gamma+1}$ as above. Furthermore, let K_{ν_j} denote the triangle with vertices v_j, v_{j+1} and w_j, $j = 1, \ldots, m$ (we set $v_{m+1} := v_1$) such that $w_j \in \hat{L}$. Then

$$K_{\nu_1} = K_{\gamma+1},$$
$$\left(\bigcup_{j=1}^{m} K_{\nu_j} \right) \setminus \hat{L} \subset \operatorname{shell}(\hat{L}) \setminus \operatorname{sm}(\hat{L}).$$

By (4.6) it follows that $\left(\bigcup_{j=1}^{m} K_{\nu_j} \right) \setminus \hat{L} \subset \tilde{K}^{\gamma+1}$. Moreover, since $\tilde{K}^{\gamma+1} = \tilde{K}^{\gamma} \cup (K_{\gamma+1} \setminus L^*)$, we have

$$K_{\nu_j} \setminus \hat{L} \subset \tilde{K}^{\gamma}, \qquad j = 2, \ldots, m.$$

As in Section 3, let us denote by z_j the unique intersection point of l_{ν_j} and $l(v_j, v_{j+1})$, $j \in \hat{I}$, where

$$\hat{I} := \{ j \in \{1, \ldots, m\} : \ l_{\nu_j} \text{ and } l(v_j, v_{j+1}) \text{ are not parallel} \}.$$

Since $\tilde{s}^{\gamma}$ has been properly defined on $\operatorname{clos} \tilde{K}^{\gamma}$, it follows that

$$\tilde{s}^{\gamma+1}(v_j) = \tilde{s}^{\gamma+1}(v_{j+1}), \qquad \text{for all} \quad j \in \{2, \ldots, m\} \setminus \hat{I},$$
$$\frac{\tilde{s}^{\gamma+1}(v_{j+1})}{\tilde{s}^{\gamma+1}(v_j)} = \varepsilon_j \frac{\rho(v_{j+1}, z_j)}{\rho(v_j, z_j)}, \qquad \text{for all} \quad j \in \hat{I} \setminus \{1\},$$

where

$$\varepsilon_j := \begin{cases} -1, & \text{if } z_j \in \operatorname{con}(v_j, v_{j+1}), \\ 1, & \text{otherwise.} \end{cases}$$

By Condition C the cycle σ is singular, *i.e.*,

$$\prod_{j \in \hat{I}} \varepsilon_j \frac{\rho(v_{j+1}, z_j)}{\rho(v_j, z_j)} = 1,$$

and, since $v_{m+1} = v_1$, we obtain

$$\frac{\tilde{s}_{\gamma+1}(v_1)}{\tilde{s}_{\gamma+1}(v_2)} = \frac{\tilde{s}^{\gamma+1}(v_{m+1})}{\tilde{s}^{\gamma+1}(v_2)} = \prod_{j=2}^{m} \frac{\tilde{s}^{\gamma+1}(v_{j+1})}{\tilde{s}^{\gamma+1}(v_j)} = \prod_{j \in \hat{I} \setminus \{1\}} \varepsilon_j \frac{\rho(v_{j+1}, z_j)}{\rho(v_j, z_j)}$$
$$= \begin{cases} \varepsilon_1 \frac{\rho(v_1, z_1)}{\rho(v_2, z_1)}, & \text{if } 1 \in \hat{I}, \\ 1, & \text{if } 1 \notin \hat{I}. \end{cases}$$

Since $\tilde{s}_{\gamma+1}$ is a linear polynomial that vanishes at v (see (4.4)), (4.5) follows. This completes the proof of the first statement of Theorem 3.4.

We now turn to the proof of the second statement of the theorem. It is obvious that

$$S_{\mathcal{L}} := \{s \in S(\Delta) : \ l_i \subset Z(s_i) \text{ for all } i \in I\},$$

is a linear subspace of $S(\Delta)$. (Here, as usual, s_i denotes the linear polynomial coinciding with $s_{|K_i}$ on K_i.) Recall that

$$K \setminus L^* = K^{[1]} \cup \ldots \cup K^{[q]},$$

where $K^{[j]}$, $j = 1, \ldots, q$, are the connected components of $K \setminus L^*$. Above we constructed the functions $s^{[j]} \in S(\Delta)$, $j = 1, \ldots, q$, satisfying (4.2). Since these functions are obviously linearly independent and belong to $S_{\mathcal{L}}$, we have

$$\dim S_{\mathcal{L}} \geq q.$$

In order to show the opposite inequality, we consider an arbitrary spline $s \in S_{\mathcal{L}}$. Observe that it can easily be seen from the construction in (III) that each $s^{[j]}$ is uniquely determined by (4.2) up to a constant factor. Therefore,

$$s_{|K^{[j]}} \equiv c_j s^{[j]}|_{K^{[j]}}, \qquad j = 1, \ldots, q,$$

for some $c_j \in \mathbb{R}$. This implies that

$$s = \sum_{j=1}^{q} c_j s^{[j]},$$

i.e., $s^{[1]}, \ldots, s^{[q]}$ form a basis for $S_{\mathcal{L}}$, and the proof is complete.

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