Locally Linearly Independent Basis for C^1 Bivariate Splines of Degree $q \geq 5$

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Abstract. We construct a locally linearly independent basis for the space $S_q^1(\Delta)$ $(q \geq 5)$. Bases with this property were available only for some subspaces of smooth bivariate splines.

§1. Introduction

Let $\Omega \subset \mathbb{R}^2$ be a simply connected polygonal domain, and let Δ denote a triangulation of Ω consisting of N triangles, V vertices and E edges. Given $0 \leq r < q$, consider the linear space of bivariate polynomial splines of degree q and smoothness r,

$$S_q^r(\Delta) := \{s \in C^r(\Omega) : \ s_{|T} \in \Pi_q \ \text{ for all triangles } T \in \Delta\},$$

where

$$\Pi_q := \operatorname{span}\{x^i y^j : \ i \geq 0, \ j \geq 0, \ i+j \leq q\}$$

is the space of bivariate polynomials of total degree q. To simplify notation, we set

$$d_q := \dim \Pi_q = \frac{(q+1)(q+2)}{2}, \quad q = 0, 1, \ldots .$$

The question of identifying the dimension of $S_q^r(\Delta)$ was first considered by Strang [14]. Morgan & Scott [11] showed that

$$\dim S_q^1(\Delta) = d_q N - (d_q - d_{q-2}) E_I + d_1 V_I + \sigma, \quad q \geq 5, \tag{1}$$

where V_I and E_I denote the number of interior vertices and interior edges respectively, and σ is the number of singular vertices of Δ, *i.e.*, those interior vertices for which the adjacent edges of each attached edge are collinear, so

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that exactly four triangles share a singular vertex and their union is a quadrilateral with the diagonals drawn in.

Moreover, in [11] a nodal basis for $S_q^1(\Delta)$ ($q \geq 5$) was constructed. This means that the functions in $S_q^1(\Delta)$ were determined by their values and derivatives at points in Ω (nodal values). An important feature of the Morgan-Scott basis is that each basis function is supported at most in the star of a vertex, *i.e.*, the union of all triangles sharing the vertex.

For many further results on the dimension and bases for the spaces $S_q^r(\Delta)$ and their superspline subspaces, see [1–5,9,10,12,13,15] and references therein.

As it has been shown recently [7], the property of *local linear independence* of a system of functions plays an important role in the problems of multivariate spline interpolation. Because of this, the question of existence of locally linearly independent systems of spline-functions was considered in [8]. Particularly, it was proved in [8, Section 3.5] that for any $\rho \geq 2r$ and $q \geq 2\rho+1$, the space of supersplines

$$S_q^{r,\rho}(\Delta) := \{s \in S_q^r(\Delta) : \; s \in C^\rho(v) \;\text{ for all vertices } v \in \Delta\}$$

admits a locally linearly independent basis. However, the Morgan-Scott basis for $S_q^1(\Delta)$ is easily seen not to be locally linearly independent (see Remark 11).

The aim of this paper is to provide a locally linearly independent basis for $S_q^1(\Delta)$, $q \geq 5$ (see Theorem 8). We note that our construction in fact differs from that of [11] only in the choice of second order derivatives at the vertices.

§2. Locally Linearly Independent Systems

Let K be a topological space, and let $F(K)$ denote the linear space of all real functions on K. We set

$$\operatorname{supp} f := \overline{\{t \in K : \; f(t) \neq 0\}}, \quad f \in F(K).$$

Definition 1. A system $\{u_1, \ldots, u_n\} \subset F(K) \setminus \{0\}$ is said to be locally linearly independent if for any $t \in K$ and any neighborhood $B(t)$ of t there exists an open set B' such that $t \in B' \subset B(t)$ and the subsystem

$$\{u_i : \; B' \cap \operatorname{supp} u_i \neq \emptyset\}$$

is linearly independent on B'.

For some other possible definitions of locally linearly independent systems of functions as well as their examples, see [7,8]. This notion turned out to be particularly important for the theory of almost interpolation.

Definition 2. Let $U \subset F(K)$ be a finite-dimensional linear space, $\dim U = n$. A set $T = \{t_1, \ldots, t_n\} \subset K$, is called an almost interpolation set with respect to U if for any system of neighborhoods B_i of t_i, $i = 1, \ldots, n$, there exist points $t_i' \in B_i$ such that $T' = \{t_1', \ldots, t_n'\}$ is admissible for Lagrange interpolation from U, *i.e.*, for any given data $\{y_1,\ldots,y_n\}$ there exists a unique function $u\in U$ satisfying

$$u(t_i') = y_i, \quad i=1,\ldots,n.$$

Theorem 3. [7] *Let $\{u_1,\ldots,u_n\}\subset F(K)$ be a locally linearly independent system and $U=\operatorname{span}\{u_1,\ldots,u_n\}$. A set $T=\{t_1,\ldots,t_n\}\subset K$, is an almost interpolation set w.r.t. U if and only if there exists some permutation σ of $\{1,\ldots,n\}$ such that*

$$t_i\in \operatorname{supp} u_{\sigma(i)}, \quad i=1,\ldots,n.$$

Another useful feature of a locally linearly independent system is that it forms a least supported basis for its span.

Theorem 4. [6] *A system $\{u_1,\ldots,u_n\}\subset F(K)\setminus\{0\}$ is locally linearly independent if and only if $\{u_1,\ldots,u_n\}$ is a least supported basis for $U=\operatorname{span}\{u_1,\ldots,u_n\}$, i.e., for every basis $\{v_1,\ldots,v_n\}$ of U there exists a permutation σ of $\{1,\ldots,n\}$ such that*

$$\operatorname{supp} u_i\subset \operatorname{supp} v_{\sigma(i)}, \quad i=1,\ldots,n.$$

The following characterization of local linear independence of a system of splines in $S_q^r(\Delta)$ is an immediate consequence of [7, Theorem 3.5].

Theorem 5. *Let $\{s_1,\ldots,s_n\}\subset S_q^r(\Delta)\setminus\{0\}$ and $\Pi_q\subset S:=\operatorname{span}\{s_1,\ldots,s_n\}$. Then $\{s_1,\ldots,s_n\}$ is locally linearly independent if and only if*

$$\operatorname{card}\{i:\ T\subset \operatorname{supp} s^{[i]}\} = d_q, \quad \textit{for every triangle } T\in\Delta. \tag{2}$$

Proof: It follows from [7, Theorems 3.5 and 3.4, 2)] that a basis for S is locally linearly independent if and only if the local dimension of S at any point t in the interior of an arbitrary triangle $T\in\Delta$ equals the number of basis functions which are supported on a neighborhood of t. Since $\Pi_q\subset S$, it is easy to see that the local dimension at such a point is always d_q and that every spline $s\in S$ which is supported on a neighborhood of t, is also supported on the whole triangle T, so that $\operatorname{card}\{i:\ T\subset\operatorname{supp} s^{[i]}\}$ equals the number of basis functions supported on a neighborhood of t. □

§3. A Basis for $S_q^1(\Delta)$ $(q\ge 5)$

Following the notation introduced in [11], we consider "edge derivatives" of splines $s\in S_q^1(\Delta)$. Let v be a vertex in Δ, let e_1,e_2 be two consecutive edges attached to v, and let T be the triangle with vertex v and edges e_1,e_2. By the first and, respectively, second e_i-derivative of s at v we mean

$$s_{e_i}(v) := \frac{\partial s}{\partial r_i}(v) \quad\text{and}\quad s_{e_i^2}(v) := \frac{\partial^2 (s_{|T})}{\partial r_i^2}(v), \quad i=1,2,$$

where r_i is the unit vector in the e_i direction away from v. Furthermore, by the **cross** (e_1, e_2)**-derivative** of s at v we mean

$$s_{e_1 e_2}(v) := \frac{\partial^2 (s_{|T})}{\partial r_1 \partial r_2}(v).$$

For every edge $e \in \Delta$ we choose one (of two possible) unit vectors orthogonal to e and denote it by $r^{\perp}$. Then the **edge normal derivative** of s at any point $z \in e$ is defined by

$$s_{e^{\perp}}(z) := \frac{\partial s}{\partial r^{\perp}}(z).$$

Let e', e, e'' be three consecutive edges attached to a vertex v. Denote by θ' and θ'' the angles between e and e' and between e and e'' respectively. Then the second e-derivative and the cross derivatives of s at v stay in the following relation

$$\sin(\theta' + \theta'') s_{e^2}(v) = \sin\theta'' s_{ee'}(v) + \sin\theta' s_{ee''}(v). \tag{3}$$

(See equation (III) in [11].) Particularly, if e is **degenerate** at v, *i.e.*, $\theta' + \theta'' = \pi$, then we have

$$\sin\theta'' s_{ee'}(v) = -\sin\theta' s_{ee''}(v). \tag{4}$$

For every vertex $v \in \Delta$, let $T_v^1, \ldots, T_v^{n_v}$ be all triangles attached to v and numbered in counterclockwise order (starting from a boundary triangle if v is a boundary vertex). Denote by e_i the common edge of T_v^{i-1} and T_v^i, $i = 2, \ldots, n_v$. If v is an interior vertex, $e_1 = e_{n_v+1}$ denote the common edge of T_v^1 and $T_v^{n_v}$. Otherwise, e_1 and e_{n_v+1} are the boundary edges (attached to v) of T_v^1 and $T_v^{n_v}$ respectively. We now consider the following set of nodal values:

1) for each vertex $v \in \Delta$, the nodal values $s(v)$, $s_x(v)$, and $s_y(v)$,
2) for each edge $e \in \Delta$, the nodal values $s(z_e^{0,1}), \ldots, s(z_e^{0,q-5})$, where $\{z_e^{0,1}, \ldots, z_e^{0,q-5}\}$ is a set of distinct points in the interior of e,
3) for each edge $e \in \Delta$, the nodal values $s_{e^{\perp}}(z_e^{1,1}), \ldots, s_{e^{\perp}}(z_e^{1,q-4})$, where $\{z_e^{1,1}, \ldots, z_e^{1,q-4}\}$ is a set of distinct points in the interior of e,
4) for each triangle $T \in \Delta$, the nodal values $s(z_T^1), \ldots, s(z_T^{d_{q-6}})$, where $\{z_T^1, \ldots, z_T^{d_{q-6}}\} \subset \operatorname{int} T$ is a set of points admissible for Lagrange interpolation from Π_{q-6},
5) for each vertex $v \in \Delta$, the nodal values $s_{e_i e_{i+1}}(v)$ for all $i \in \{1, \ldots, n_v\}$ such that e_i is nondegenerate at v,
6) for each vertex $v \in \Delta$, the nodal values $s_{e_i^2}(v)$ for all i such that e_i is degenerate at v,
7) for each boundary vertex $v \in \Delta$, the nodal values $s_{e_1^2}(v)$ and $s_{e_{n_v+1}^2}(v)$,
8) for each singular vertex $v \in \Delta$, the nodal value $s_{e_1 e_2}(v)$.

While our sets of nodal values of type 1–4 are the same as those of the paper [11], the remaining nodal values are chosen differently. However, it is easy to see that the number of nodal values in 5)–8) is n_v if v is a nonsingular interior vertex, $n_v + 1$ if v is a singular vertex, and $n_v + 2$ if v is a boundary vertex, which equals the number of nodal values of type 5 and 6 in [11]. Therefore, a calculation in [11] shows that the total number of nodal values listed in 1)–8) is equal to the dimension of $S_q^1(\Delta)$.

Lemma 6. *Suppose that $s \in S_q^1(\Delta)$ has all nodal values in 1)–8) equal to zero. Then $s \equiv 0$.*

Proof: Following the argumentation in [11], we only need to show that for each vertex $v \in \Delta$,

$$s_{e_i^2}(v) = 0, \quad i = 1, \ldots, n_v + 1, \quad \text{and} \quad s_{e_i e_{i+1}}(v) = 0, \quad i = 1, \ldots, n_v.$$

Let us first consider cross derivatives. Since the nodal values of type 5 are zero, we have $s_{e_i e_{i+1}}(v) = 0$ for all i such that e_i is nondegenerate at v. If e_i is degenerate at v, then it follows from (4) that

$$\frac{1}{\sin\theta_i} s_{e_i e_{i+1}}(v) = -\frac{1}{\sin\theta_{i-1}} s_{e_{i-1} e_i}(v),$$

where θ_i denotes the angle between e_i and e_{i+1}. If now e_{i-1} is nondegenerate at v, then $s_{e_{i-1} e_i}(v) = 0$ and hence $s_{e_i e_{i+1}} = 0$. If both e_i and e_{i-1} are degenerate at v, but e_{i-2} is nondegenerate, then

$$\frac{1}{\sin\theta_i} s_{e_i e_{i+1}}(v) = -\frac{1}{\sin\theta_{i-1}} s_{e_{i-1} e_i}(v) = \frac{1}{\sin\theta_{i-2}} s_{e_{i-2} e_{i-1}}(v) = 0,$$

hence $s_{e_i e_{i+1}} = 0$. Finally, if at least three edges are degenerate at v, then v is necessarily a singular vertex, $n_v = 4$ and all four edges e_1, e_2, e_3 and e_4 are degenerate at v. Then

$$\frac{1}{\sin\theta_1} s_{e_1 e_2}(v) = -\frac{1}{\sin\theta_2} s_{e_2 e_3}(v) = \frac{1}{\sin\theta_3} s_{e_3 e_4}(v) = -\frac{1}{\sin\theta_4} s_{e_4 e_1}(v).$$

Since the nodal value of type 8 is zero, we deduce from the last equation that all four cross derivatives at v are zero.

It remains to show that all second e_i-derivatives are also zero at v. Since the nodal values of type 6 and 7 are zero, we only consider those i for which e_i lies in the interior of Ω and is nondegenerate at v. Then $\sin(\theta_i + \theta_{i-1}) \neq 0$, so that by (3),

$$s_{e_i^2}(v) = \frac{\sin\theta_i}{\sin(\theta_i + \theta_{i-1})} s_{e_i e_{i+1}}(v) + \frac{\sin\theta_{i-1}}{\sin(\theta_i + \theta_{i-1})} s_{e_i e_{i-1}}(v).$$

We have already proved that $s_{e_i e_{i+1}}(v) = s_{e_i e_{i-1}}(v) = 0$. Therefore, $s_{e_i^2}(v) = 0$. $\square$

Let us number the nodal values $1, \dots, D$, where

$$D := \dim S_q^1(\Delta) = d_q N - (d_q - d_{q-2}) E_I + d_1 V_I + \sigma.$$

Then, in view of Lemma 6, it follows from basic linear algebra that for each $j \in \{1, \dots, D\}$ there exists a unique spline $s^{[j]} \in S_q^1(\Delta)$ which has the j-th nodal value equal to 1 and all the other nodal values zero. Moreover, it is clear that $\{s^{[1]}, \dots, s^{[D]}\}$ is a basis for $S_q^1(\Delta)$. We say that a basis function $s^{[j]}$ is of type 1–7 or 8 if the j-th nodal value is of the corresponding type.

We now describe the supports of the basis functions $s^{[j]}$.

Lemma 7.

1) *The support of a basis function $s^{[j]}$ of type 1 is the star of the vertex v.*
2) *The support of a basis function $s^{[j]}$ of type 2 is the union of the two triangles sharing the edge e.*
3) *The support of a basis function $s^{[j]}$ of type 3 is the union of the two triangles sharing the edge e.*
4) *The support of a basis function $s^{[j]}$ of type 4 is the triangle T.*
5) *The support of a basis function $s^{[j]}$ of type 5 is either the union of the triangles T_v^{i-1}, T_v^i and T_v^{i+1} if the edge e_{i+1} is nondegenerate at v, or the union of the triangles T_v^{i-1}, T_v^i, T_v^{i+1} and T_v^{i+2} if the edge e_{i+1} is degenerate at v, but e_{i+2} is nondegenerate at v, or the union of the triangles T_v^{i-1}, T_v^i, T_v^{i+1}, T_v^{i+2} and T_v^{i+3} if the edges e_{i+1} and e_{i+2} both are degenerate at v, but e_{i+3} is nondegenerate at v. (If v is a boundary vertex, then the triangles with a superscript not in $\{1, \dots, n_v\}$ should be omitted.)*
6) *The support of a basis function $s^{[j]}$ of type 6 is the union of the two triangles sharing the edge e_i.*
7) *The support of a basis function $s^{[j]}$ of type 7 is one of the triangles T_v^1 or $T_v^{n_v}$.*
8) *The support of a basis function $s^{[j]}$ of type 8 is the star of the vertex v.*

Proof: The statements 1)–4) are obvious. In order to prove 5)–8), we argue in the same way as in the proof of Lemma 6, except that one of the nodal values of $s \in S_q^1(\Delta)$ is 1 while the others are zero. Suppose, for example, that $s^{[j]}$ is of type 5 and the corresponding nodal value is $s_{e_\mu e_{\mu+1}}(v)$, with both e_μ and $e_{\mu+1}$ nondegenerate at v. Then $s^{[j]}_{e_\mu e_{\mu+1}}(v) = 1$ and all the other nodal values of $s^{[j]}$ are zero. The same calculation as in the proof of Lemma 6 shows that

$$\begin{aligned} s^{[j]}_{e_i^2}(v) &= 0, \quad i = 1, \dots, n_v + 1, \quad i \notin \{\mu, \mu+1\}, \\ s^{[j]}_{e_i e_{i+1}}(v) &= 0, \quad i = 1, \dots, n_v, \quad i \neq \mu, \end{aligned}$$

and

$$s^{[j]}_{e_\mu^2}(v) = \frac{\sin\theta_\mu}{\sin(\theta_\mu + \theta_{\mu-1})} \neq 0, \quad s^{[j]}_{e_{\mu+1}^2}(v) = \frac{\sin\theta_\mu}{\sin(\theta_\mu + \theta_{\mu+1})} \neq 0,$$

from which it follows that $\operatorname{supp} s^{[j]} = T_v^{\mu-1} \cup T_v^{\mu} \cup T_v^{\mu+1}$. $\square$

As soon as Lemma 7 is established, it is easy to check that the basis $\{s^{[1]}, \dots, s^{[D]}\}$ satisfies (2), so that the following theorem is valid.

Theorem 8. *The above-constructed basis* $\{s^{[1]}, \dots, s^{[D]}\}$ *for* $S_q^1(\Delta)$ $(q \geq 5)$ *is locally linearly independent.*

In view of Theorems 3 and 4, the next two statements immediately follow from Theorem 8.

Corollary 9. *Let* $\{t^{[1]}, \dots, t^{[D]}\} \subset \Omega$ *be a set of distinct points such that* $t^{[j]} \in \operatorname{supp} s^{[j]}$, $j = 1, \dots, D$, *where* $\operatorname{supp} s^{[j]}$ *are described in Lemma 7. Then* $\{t^{[1]}, \dots, t^{[D]}\}$ *is an almost interpolation set with respect to* $S_q^1(\Delta)$.

Corollary 10. $\{s^{[1]}, \dots, s^{[D]}\}$ *is a least supported basis for* $S_q^1(\Delta)$.

Remark 11. The original scheme by Morgan and Scott makes use of one cross derivative and all except one edge derivatives at each nonsingular interior vertex v. Let us denote the exceptional edge by e. It is not difficult to see that all basis functions corresponding to second order derivatives at v are supported on both triangles T' and T'' sharing e. Hence, there are at least $d_q + n_v - 3$ basis functions supported on T', where n_v is the number of triangles attached to v. By Theorem 5 it then follows that the Morgan-Scott basis fails to be locally linearly independent as soon as there exists a nonsingular interior vertex v with $n_v > 3$.

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