Approximation Order of Bivariate Spline Interpolation for Arbitrary Smoothness

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Abstract

By using the algorithm of Nürnberger & Riessinger [11], we construct Hermite interpolation sets for spaces of bivariate splines $S_q^r(\Delta^1)$ of arbitrary smoothness defined on the uniform type triangulations. It is shown that our Hermite interpolation method yields optimal approximation order for $q \geq 3.5r + 1$. In order to prove this, we use the concept of weak interpolation and arguments of Birkhoff interpolation.

Introduction

We investigate spline spaces of the following type. Let a rectangle R and a partition of R into uniform subrectangles be given. We add to each subrectangle the same diagonal and denote the resulting partition by Δ^1. (If we add both diagonals, then the resulting partition is denoted by Δ^2.) The space of bivariate splines of degree q and smoothness r with respect to the partition Δ^i is denoted by $S^r_q(\Delta^i),\ i = 1, 2$.

Nürnberger and Riessinger [10], [11] developed a method for constructing point sets which admit unique Lagrange interpolation from $S^r_q(\Delta^i),\ i = 1, 2$.

The aim of this paper is to define appropriate Hermite interpolation sets which can be considered as a limit case of the Lagrange interpolation sets above and to show that the corresponding interpolating splines yield optimal approximation order for $S^r_q(\Delta^1)$ if $q \geq 3.5r + 1$. More precisely, for each $f \in C^{q+1}(R)$ the interpolating spline $s_f \in S^r_q(\Delta^1)$ satisfies $\|D^\omega(f - s_f)\| \leq Kh^{q+1-\omega}$ for $\omega \in \{0, \ldots, q\}$. Here h denotes the maximal sidelength of the subrectangles of the partition and the constant $K > 0$ is independent of h.

Nürnberger [9] showed that the interpolating spline s_f yields (nearly) optimal approximation order for $S^1_q(\Delta^1),\ q \geq 4$, which means $\|D^\omega(f - s_f)\| \leq Kh^{\rho-\omega}$ for $\omega \in \{0, \ldots, \rho - 1\}$, where $\rho = 4$ if $q = 4$ and $\rho = q + 1$ if $q \geq 5$. Nürnberger & Walz [12] proved that the interpolating spline s_f yields (nearly) optimal approximation order for $S^1_q(\Delta^2),\ q \geq 2$, which means $\|D^\omega(f - s_f)\| \leq Kh^{\rho-\omega}$ for $\omega \in \{0, \ldots, \rho-1\}$, where $\rho = q$ if $q \in \{2, 3\}$ and $\rho = q + 1$ if $q \geq 4$. The interpolating splines in $S^r_q(\Delta^i),\ i = 1, 2$ can be computed locally by passing from one triangle to the next and by solving small systems (see the numerical examples in [9], [11], [12]).

Our proof of the optimal approximation order for $S^r_q(\Delta^1),\ q \geq 3.5r + 1$, is based on developements of the concept of weak interpolation introduced in [9]. In order to prove that there exists $dim\ \tilde{\Pi}_q$ weak interpolation conditions on each triangle we have to develop new arguments. This is done by showing how weak interpolation conditions are transfered across the edges and by applying Birkhoff interpolation methods for univariate polynomials.

We remark that the order of $dist(f, S^r_q(\Delta))$ is optimal, if $q \geq 3r + 2$ (see de Boor & Höllig [1], Chui, Hong & Jia [4], Lai & Schumaker [8]). On the other hand, it was shown by de Boor & Jia [2] that this is not true for $q < 3r + 2$, even for the three-directional mesh. We finally note that our method is different from Bernstein-Bezier techniques for interpolation by $S^r_q(\Delta^i),\ i = 1, 2$, used for $r = 1$ and $q \leq 3$ in [3], [5], [13], [14], [15].

Main Results

We consider bivariate splines of the following type. First, the ***space of bivariate***

polynomials of total degree q is denoted by

$$\tilde{\Pi}_q = \text{span}\{x^\alpha y^\beta : \ \alpha \geq 0, \ \beta \geq 0, \ \alpha + \beta \leq q\} \ .$$

(The corresponding univariate polynomial space is denoted by Π_q.) Let a rectangle $R = [a_0, b_0] \times [c_0, d_0]$ and points $a_0 = x_0 < x_1 < \ldots < x_{n_1-1} < x_{n_1} = b_0, \ c_0 = y_0 < y_1 < \ldots < y_{n_2-1} < y_{n_2} = d_0$ such that $x_i - x_{i-1} = h_1, \ i = 1, \ldots, n_1$ and $y_j - y_{j-1} = h_2, \ j = 1, \ldots, n_2$ be given. By defining $R_{i,j} = (x_{i-1}, x_i) \times (y_{j-1}, y_j), \ i = 1, \ldots, n_1, \ j = 1, \ldots, n_2$, we obtain a partition of R into subrectangles $R_{i,j}$. We set $z_{i,j} = (x_i, y_j), \ i = 0, \ldots, n_1, \ j = 0, \ldots, n_2$, add the diagonal from $z_{i-1,j-1}$ to $z_{i,j}$ to each subrectangle $R_{i,j}$ and denote by $T_{i,j}^{(1)}$ (respectively $T_{i,j}^{(2)}$) the upper left (respectively the lower right) triangle of $R_{i,j}$. The resulting partition is Δ^1.

The space $S_q^r(\Delta^1)$ is defined as follows. Let integers r and q with $0 \leq r < q$ be given. The space $S_q^r(\Delta^1)$ of all functions $s \in C^r(R)$ such that the restriction to each subtriangle of the partition Δ^1 is in $\tilde{\Pi}_q$ is called space of ***bivariate splines of degree q and smoothness r***.

We now investigate interpolation by $S_q^r(\Delta^1)$. In contrast to the univariate case, it is a non-trivial problem to construct any set at which interpolation by $S_q^r(\Delta^1)$ is possible. Therefore, we formulate the following problem: Determine a set $\{z_1, \ldots, z_N\}$ in R, where $N = dim \ S_q^r(\Delta^1)$ such that for each function $f \in C(R)$, the ***Lagrange interpolation problem*** $s_f(z_i) = f(z_i), \ i = 1, \ldots, N$ has a unique solution $s_f \in S_q^r(\Delta^1)$. Such a set $\{z_1, \ldots, z_N\}$ is called ***Lagrange interpolation set*** for $S_q^r(\Delta^1)$.

If we consider not only function values of f but also partial derivatives of a sufficiently differentiable f, then we speak of a ***Hermite interpolation problem*** for the space $S_q^r(\Delta^1)$, and the corresponding sets are called ***Hermite interpolation sets*** for $S_q^r(\Delta^1)$. For describing Hermite interpolation conditions, we denote by f_x and f_y the partial derivatives of f for x and y, respectively. The higher partial derivatives are denoted by $f_{x^\alpha y^\beta}$. Given a point $z = (x, y) \in R$, we set

$$D^\omega f(z) = (f_{x^\omega}(z), f_{x^{\omega-1}y}(z), \ldots, f_{xy^{\omega-1}}(z), f_{y^\omega}(z)) \ .$$

The uniform norm of f is defined by $\|f\| = \max\{|f(z)| : \ z \in R\}$ and for the derivatives, we set

$$\|D^\omega f\| = \max\{\|f_{x^\alpha y^\beta}\| : \ \alpha \geq 0, \ \beta \geq 0, \ \alpha + \beta = \omega\} \ .$$

In the following, we construct Hermite interpolation sets for $S_q^r(\Delta^1)$. The construction of Hermite interpolation sets is done by describing Lagrange interpolation sets for these spaces and then "taking limits". The following construction of Lagrange interpolation sets is a special case of [10], [11].

Construction of Lagrange interpolation sets

In order to construct Lagrange interpolation sets for $S_q^r(\Delta^1)$, we only have to describe four basic steps. For an arbitrary subtriangle T of the partition Δ^1, one of the following four steps will be applied on T.

Step A. (Starting step) Choose $q+1$ disjoint line segments $a_1, \ldots, a_{q+1}$ in T. For $i = 1, \ldots, q+1$, choose $q+2-i$ distinct points on a_i.

Step B. Choose $q-r$ disjoint line segments $b_1, \ldots, b_{q-r}$ in T. For $i = 1, \ldots, q-r$, choose $q+1-r-i$ distinct points on b_i.

Step C. Choose $q-2r+[\frac{r}{2}]$ disjoint line segments $c_1, \ldots, c_{q-2r+[\frac{r}{2}]}$ in T. For $i = 1, \ldots, q-2r$, choose $q+1-r-i$ distinct points on c_i and for $i = q-2r+1, \ldots, q-2r+[\frac{r}{2}]$ choose $2(q-i)-3r+1$ distinct points on c_i.

Step D. Choose $q-2r-1$ disjoint line segments $d_1, \ldots, d_{q-2r-1}$ in T. For $i = 1, \ldots, q-2r-1$, choose $q-2r-i$ distinct points on d_i.

Here and in the following, we set $[b] = \max\{a \in \mathbb{Z} : \ a \leq b\}$. If the number of lines in step C or step D is non-positive, then no points are chosen.

Given a partition Δ^1, we construct interpolation sets by applying the above steps successively to the subtriangles. We choose diagonal (respectively horizontal) line segments in $T_{i,j}^{(1)}$ (respectively $T_{i,j}^{(2)}$); except in the first triangle in the upper row, $T_{1,n_2}^{(1)}$, where we choose horizontal line segments (see Figure 2). The points chosen on these line segments shall not lie on the triangles already considered. First, we apply step A to $T_{1,n_2}^{(1)}$ (starting triangle). Then, by passing from the left to the right, we apply step B to the triangles $T_{i,n_2}^{(k)}$, $(i,k) \in \{(i_1,k_1) : \ i_1 = 1, \ldots, n_1, \ k_1 = 1,2\} \setminus \{(1,1)\}$. Then we consider $j = n_2 - 1$. We apply step B to $T_{1,n_2-1}^{(1)}$ and $T_{n_1,n_2-1}^{(2)}$. By passing from the left to the right, we apply step C (respectively step D) to the triangles $T_{i,n_2-1}^{(2)}$, $i = 1, \ldots, n_1 - 1$ (respectively $T_{i,n_2-1}^{(1)}$, $i = 2, \ldots, n_2$). Then we consider the next row and apply the same steps as for $j = n_2 - 1$. We continue this method until all rows of the partition are considered (see Figure 1).

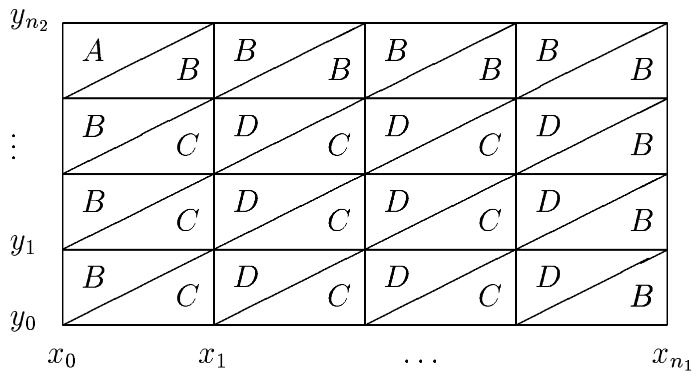

Figure 1: Interpolation steps for $S_q^r(\Delta^1)$

Next, we construct Hermite interpolation sets for $S_q^r(\Delta^1)$. This is done by using the Lagrange interpolation above and by "taking limits". We consider the Lagrange configurations and let certain points and line segments coincide (Figure 2 indicates which points and line segments shall coincide). If certain points on some line segments coincide, then we pass to the directional derivatives orthogonal to the line segments. In this way, we obtain the following Hermite interpolation problem.

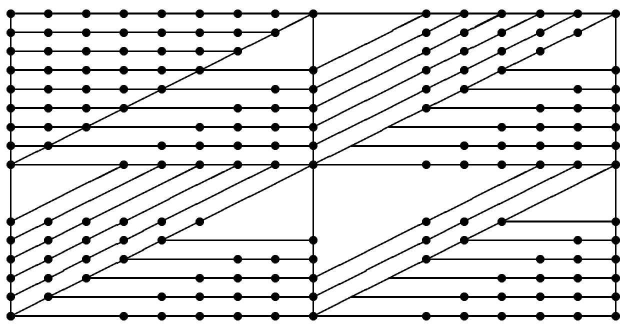

Figure 2: Interpolation set for $S_8^2(\Delta^1)$

Construction of Hermite interpolation sets

Let a sufficiently differentiable function $f \in C(R)$ be given. For defining Hermite interpolation conditions for a spline $s \in S_q^r(\Delta^1)$, we only have to describe four basic conditions. Let T be an arbitrary subtriangle of the partition Δ^1. If T is not the first from the left triangle in the top row, then $\tilde{T}$ denotes the adjacent subtriangle left of T in the same row if it exists and up of T otherwise. One of the following four conditions will be imposed on the polynomial $p = s|_T \in \tilde{\Pi}_q$.

Condition A. (Starting condition) $D^\omega p(z) = D^\omega f(z),\ \omega = 0, \ldots, q$, where z is a vertex of T.

Condition B. $D^\omega p(z) = D^\omega f(z),\ \omega = 0, \ldots, q-r-1$, where z is the vertex of T not belonging to $\tilde{T}$.

Condition C. $p_{x^\alpha y^\beta}(z) = f_{x^\alpha y^\beta}(z),\ \alpha \geq 0,\ \beta \geq 0,\ \alpha+\beta \leq q-r-1,\ \alpha+2\beta \leq 2q-3r-2$, where z is the vertex of T not belonging to $\tilde{T}$.

Condition D. $D^\omega p(z) = D^\omega f(z),\ \omega = 0, \ldots, q-2r-2$, where z is the midpoint of the diagonal of T.

While Condition A,B and D are symmetric with respect to x and y, this is not the fact with condition C. Figure 3 presents the domain in which all integer points (α, β) are taken in order to get condition C.

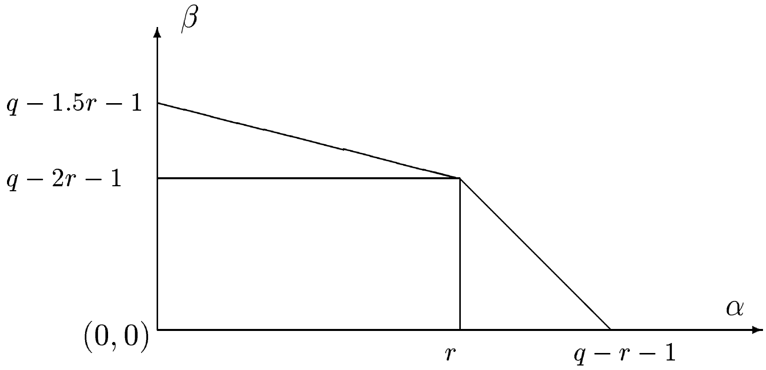

Figure 3: Condition C

Given a partition Δ^1, we impose interpolation conditions on s by passing from the upper to the lower row and by passing from the first to the last triangle in each row as follows (see Figure 1).

First, we assign condition A to $T^{(1)}_{1,n_2}$. Then by passing from the left to the right, we assign condition B to the remaining triangles of the upper row. Then we consider $j = n_2 - 1$. We assign condition B to the lower vertex of the first triangle in this row. Then, by passing from the left to the right, we alternatingly assign condition C and condition D to the remaining triangles in the row, except that to the last triangle we assign condition B. Then we consider the next row and assign the same conditions as in the row before. We continue this method until all rows of the partition are considered. (Note that the order of the condition in the starting row is different from the conditions in all other rows.)

We prove our main theorem on approximation order (Theorem 2) by showing that the interpolating spline satisfies $dim\tilde{\Pi}_q$ weak interpolation conditions on each subtriangle of Δ^1 and then apply Lemma 3 below. By using even simpler arguments as in the proof of Theorem 2, it follows that a spline $s \in S_q^r(\Delta^1)$ which fulfills the homogeneous Hermite interpolation conditions above satisfies $dim\tilde{\Pi}_q$ homogeneous interpolation conditions on each subtriangle of Δ^1. Therefore, we get the following result.

Theorem 1. *For each sufficiently differentiable function $f \in C(R)$, there exists a unique spline $s_f \in S_q^r(\Delta^1)$, $q \geq 3.5r + 1$, which satisfies the Hermite interpolation conditions described above.*

The next result shows that the Hermite interpolation method described above yields optimal approximation order for $q \geq 3.5r + 1$. We denote by ϕ the angle between the horizontal and the diagonal lines of the partition Δ^1 and set $h = \max\{h_1, h_2\}$. In the following theorem, the norm denotes the maximum of the uniform norm over all subtriangles of the partition (w.r.t. the polynomial pieces).

Theorem 2. *Let $q,\ r \in \mathbb{N} \cup \{0\}$ be given such that $q \geq 3.5r + 1$. For each function $f \in C^{q+1}(R)$, there exists a constant $K > 0$, such that for the unique interpolating spline $s_f \in S_q^r(\Delta^1)$ in Theorem 1 and for all $\omega \in \{0, \ldots, q\}$,*

$$\|D^\omega(f - s_f)\| \leq K h^{q+1-\omega} \ . \tag{1}$$

(The constant $K > 0$ depends on r, q, ϕ, $\|D^{q+1}f\|$ and is independent of h.)

In order to prove Theorem 2, we need the following result on weak interpolation by bivariate polynomials. Let a triangle W with vertices $(0,0)$, $(\lambda_1, 0)$ and (λ_2, λ_3), where $\lambda_3 > 0$ be given. Moreover, let $0 \leq y_0 \leq y_1 \leq \ldots \leq y_{q-1} \leq y_q \leq \lambda_3$ and for each $j \in \{0, \ldots, q\}$, $x_{0,j} \leq \ldots \leq x_{q-j,j}$ be given such that all points $z_{i,j} = (x_{i,j}, y_j)$, $i = 0, \ldots, q-j$, $j = 0, \ldots, q$ are contained in W. To each point $z_{i,j}$, we assign integers

$$\alpha_{i,j} = \max\{\alpha : \ x_{i-\alpha,j} = \ldots = x_{i,j}\} \ \text{ and } \ \beta_j = \max\{\beta : \ y_{j-\beta} = \ldots = y_j\} \ .$$

The next result on weak interpolation follows from Lemma 4 in [9].

Lemma 3. *Let a function $f \in C^{q+1}(W)$ and a family of bivariate polynomials $\{p_h \in \tilde{\Pi}_q : \ h \in (0,1]\}$ be given. Suppose that there exists a constant $\tilde{K} > 0$ such that for all $h \in (0,1]$,*

$$|(f - p_h)_{x^{\alpha_{i,j}} y^{\beta_j}}(h z_{i,j})| \leq \tilde{K} h^{q+1-\alpha_{i,j}-\beta_j}, \qquad i = 0, \ldots, q-j,\ j = 0, \ldots, q \ . \tag{2}$$

Then there exists a constant $K > 0$ such that for all $h \in (0,1]$ and $\omega \in \{0, \ldots, q\}$,

$$\|D^\omega(f - p_h)\|_{hW} \leq K h^{q+1-\omega} \ . \tag{3}$$

*The constant K depends on $\tilde{K}$, q, $\|D^{q+1}f\|$, the smallest angle of W and is independent of h. (We briefly say that p_h **weakly interpolates** f on hW if (2) is satisfied.)*

For the proof of Theorem 2, we also need a result on weak interpolation by a set of univariate polynomials $\{g_h \in \Pi_q : \ h \in (0,1]\}$ in the sense of Birkhoff interpolation. Therefore, let $0 \leq t_0 \leq \ldots \leq t_q \leq 1$ and integers $\gamma_j \in \{0,\ldots,q\}$, $j = 0,\ldots,q$, be given such that if $i \neq j$ and $t_i = t_j$, then $\gamma_i \neq \gamma_j$. We say that the ***Birkhoff interpolation problem is well-posed*** for Π_q if for each sufficiently differentiable function $f \in C[0,1]$, there exists a unique polynomial $g \in \Pi_q$ with the following properties:

$$g^{(\gamma_j)}(t_j) = f^{(\gamma_j)}(t_j), \qquad j = 0,\ldots,q \; . \tag{4}$$

Lemma 4. *Let a function $f \in C^{q+1}[0,1]$ and a family of univariate polynomials $\{g_h \in \Pi_q : \ h \in (0,1]\}$ be given. Suppose that the Birkhoff interpolation is well-posed and that there exists a constant $\tilde{C} > 0$ such that for all $h \in (0,1]$,*

$$|(f - g_h)^{(\gamma_j)}(ht_j)| \leq \tilde{C} h^{q+1-\gamma_j}, \qquad j = 0,\ldots,q \; . \tag{5}$$

Then there exists a constant $C > 0$ such that for all $h \in (0,1]$ and $\omega \in \{0,\ldots,q\}$,

$$\|(f - g_h)^{(\omega)}\|_{[0,h]} \leq C h^{q+1-\omega} \; . \tag{6}$$

The constant C depends on $\tilde{C}$, q, $\|f^{q+1}\|$ and is independent of h. (We briefly say that g_h ***weakly interpolates*** *f on $[0,h]$ if (5) is satisfied.)*

Proof. Let $\tilde{g} \in \Pi_q$ be the Taylor polynomial of f, defined by

$$\tilde{g}(t) = \sum_{\mu=0}^{q} \frac{f^{(\mu)}(0)}{\mu!} t^\mu, \qquad t \in [0,1] \; .$$

It is well known that for $\omega \in \{0,\ldots,q\}$,

$$|(f - \tilde{g})^{(\omega)}(ht)| \leq \frac{\|f^{(q+1)}\|}{(q+1-\omega)!} h^{q+1-\omega}, \qquad t \in [0,1] \; .$$

It follows that there exists a constant $C_1 > 0$ such that for all $h \in (0,1]$ and $\omega \in \{0,\ldots,q\}$,

$$\|(f - \tilde{g})^{(\omega)}\|_{[0,h]} \leq C_1 h^{q+1-\omega} \; , \tag{7}$$

where C_1 depends on q, $\|f^{(q+1)}\|$ and is independent of h. Thus, we obtain for $\omega \in \{0,\ldots,q\}$,

$$\begin{aligned} \|(f - g_h)^{(\omega)}\|_{[0,h]} &\leq \|(f - \tilde{g})^{(\omega)}\|_{[0,h]} + \|(g_h - \tilde{g})^{(\omega)}\|_{[0,h]} \\ &\leq C_1 h^{q+1-\omega} + \|(g_h - \tilde{g})^{(\omega)}\|_{[0,h]} \; . \end{aligned} \tag{8}$$

We set $\hat{g}_h = g_h - \tilde{g} \in \Pi_q$, and have to show that there exists a constant $C_2 > 0$ (independent of h) such that for all $h \in (0,1]$ and $\omega \in \{0,\ldots,q\}$,

$$\|\hat{g}_h^{(\omega)}\|_{[0,h]} \leq C_2 h^{q+1-\omega} \ . \tag{9}$$

Since Birkhoff interpolation is well-posed, the polynomial $\hat{g}_h \in \Pi_q$ can be written as follows

$$\hat{g}_h \equiv \sum_{i=0}^{q} \hat{g}_h^{(\gamma_i)}(ht_i) l_{h,i} \ , \tag{10}$$

where $l_{h,i} \in \Pi_q, \ i = 0,\ldots,q$, are the fundamental polynomials defined by the conditions

$$l_{h,i}^{(\gamma_j)}(ht_j) = \delta_{i,j}, \qquad j = 0,\ldots,q \ .$$

(Here, $\delta_{i,j}$ denotes the Kronecker symbol.) For all $\omega \in \{0,\ldots,q\}$ and $t \in [0,h]$, we have

$$l_{h,i}^{(\omega)}(t) = h^{\gamma_i-\omega} l_{1,i}^{(\omega)}(\frac{t}{h}), \qquad i = 0,\ldots,q \ , \tag{11}$$

which immediately follows from the fact that the polynomial $h^{\gamma_i} l_{1,i}(\frac{\cdot}{h}) \in \Pi_q$ satifies the same interpolation conditions as $l_{h,i}$. ¿From (11), we obtain for all $h \in (0,1]$ and $\omega \in \{0,\ldots,q\}$,

$$\|l_{h,i}^{(\omega)}\|_{[0,h]} = h^{\gamma_i-\omega} \|l_{1,i}^{(\omega)}\|_{[0,1]}, \qquad i = 0,\ldots,q \ . \tag{12}$$

It follows from (5) and (7) that for all $h \in (0,1]$ and $j \in \{0,\ldots,q\}$,

$$|\hat{g}_h^{(\gamma_j)}(ht_j)| \leq |(f-g_h)^{(\gamma_j)}(ht_j)| + |(f-\tilde{g})^{(\gamma_j)}(ht_j)| \leq (\tilde{C}+C_1) h^{q+1-\gamma_j} \ . \tag{13}$$

Thus, we get from (10), (12) and (13) that for all $h \in (0,1]$ and $\omega \in \{0,\ldots,q\}$,

$$\|\hat{g}_h^{(\omega)}\|_{[0,h]} \leq (\tilde{C}+C_1) \sum_{i=0}^{q} \|l_{1,i}^{(\omega)}\|_{[0,1]} h^{q+1-\gamma_i} h^{\gamma_i-\omega} \ .$$

By denoting $C_2 = (\tilde{C}+C_1) \sum_{i=0}^{q} \|l_{1,i}^{(\omega)}\|_{[0,1]}$, we obtain (9) which completes the proof. ■

Remark 5. For the particular choice $\gamma_j = \max\{\gamma : \ t_{j-\gamma} = \ldots = t_j\}, \ j = 0,\ldots,q$, condition (4) reduces to a Hermite interpolation condition. This generalizes the concept of univariate weak interpolation introduced in [9].

Remark 6. In the proof of Theorem 2, we will apply Lemma 4 to certain families of derivatives $\{(p_h)_{(l^\perp)^\gamma} \in \tilde{\Pi}_{q-\gamma} : \ h \in (0,1]\}$ in direction of $l^\perp$ restricted to $I_l = \{tl : \ t \in$

$[0,1]\}$, where $l=(l_1,l_2)$ is a unit vector, $l^\perp=(-l_2,l_1)$ and $\{p_h\in\tilde{\Pi}_q:\ h\in(0,1]\}$ is a family of bivariate polynomials.

In order to prove Theorem 2, we need the following result on transfering weak interpolation conditions for splines from one triangle to the next. Let $z\in\mathbb{R}^2$ and $l=(l_1,l_2)$, $m=(m_1,m_2)$ be two lineary independent unit vectors which are not orthogonal and let the cones

$$\begin{aligned} D_1 &= \{z+\tau_1 m+\tau_2 m^\perp:\ \tau_1,\ \tau_2\ge 0\}\ , \\ D_2 &= \{z+\tau_1 l+\tau_2 m^\perp:\ \tau_1,\ \tau_2\ge 0\}\ , \end{aligned}$$

such that $z+l\notin D_1$ and $z+m\notin D_2$ be given (see Figure 4).

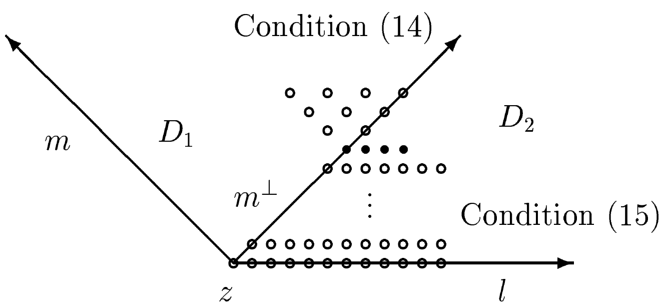

Figure 4 : Transfering weak interpolation conditions (r=3)

The following result on transfering weak interpolation conditions for a family of bivariate splines holds. We illustrate condition (14) and (15) of the following lemma in Figure 4.

Lemma 7. *Let a function $f\in C^{q+1}(D_1\cup D_2)$, a family of bivariate splines $\{s_h\in C^r(D_1\cup D_2):\ s_h|_{D_i}=p_{h,i}\in\tilde{\Pi}_q,\ i=1,2,\ \ h\in(0,1]\}$ and an integer $k\in\{0,\dots,q-r-1\}$ be given. If there exist constants $K_1,\ K_2>0$ such that for all $h\in(0,1]$,*

$$|(f-p_{h,1})_{m^\alpha(m^\perp)^\beta}(z)|\le K_1h^{q+1-\alpha-\beta},\qquad \alpha+\beta\le r+k+1,\ \alpha\ge 0,\ \beta\ge k+1 \tag{14}$$
$$|(f-p_{h,2})_{(l^\perp)^\alpha l^\beta}(z)|\le K_2h^{q+1-\alpha-\beta},\qquad \beta=0,\dots,q-\alpha,\ \alpha=0,\dots,k\ , \tag{15}$$

then there exists a constant $K_3>0$ such that for all $h\in(0,1]$ and $\beta\in\{0,\dots,r\}$,

$$|(f-p_{h,2})_{(l^\perp)^{k+1}l^\beta}(z)|\le K_3h^{q-k-\beta}\ . \tag{16}$$

Proof. First, we show that for all $h\in(0,1]$ and $\beta\in\{0,\dots,r\}$,

$$|(f-p_{h,2})_{(m^\perp)^{k+1}l^\beta}(z)|\ \ \le\ \ K_4h^{q-k-\beta}, \tag{17}$$

for some constant $K_4 > 0$. Here and in the following, we use the formula

$$F_{(\alpha_1 d_1+\alpha_2 d_2)^\lambda} \equiv \sum_{\mu=0}^{\lambda} \binom{\lambda}{\mu} \alpha_1^{\lambda-\mu} \alpha_2^{\mu} F_{d_1^{\lambda-\mu} d_2^{\mu}} \ , \ F \in C^\lambda(D_i) \ , \tag{18}$$

where d_1, d_2 and $\alpha_1 d_1 + \alpha_2 d_2$ are unit vectors and λ is a natural number. Setting $\alpha_1 = \langle l, m\rangle$ and $\alpha_2 = \langle l, m^\perp\rangle$, we get $l = \alpha_1 m + \alpha_2 m^\perp$. (Here, $\langle ., .\rangle$ denotes the standard inner product.) Therefore, by (18),

$$(f - p_{h,2})_{(m^\perp)^{k+1} l^\beta}(z) = \sum_{\mu=0}^{\beta} \binom{\beta}{\mu} \alpha_1^{\beta-\mu} \alpha_2^{\mu} (f - p_{h,2})_{m^{\beta-\mu}(m^\perp)^{k+1+\mu}}(z) \ , \tag{19}$$

for all $\beta \in \{0, \ldots, r\}$. Since $\beta - \mu \le r$, and because of the C^r-property of s_h,

$$(f - p_{h,2})_{m^{\beta-\mu}(m^\perp)^{k+1+\mu}}(z) = (f - p_{h,1})_{m^{\beta-\mu}(m^\perp)^{k+1+\mu}}(z), \qquad \mu = 0, \ldots, \beta \ , \tag{20}$$

for all $\beta \in \{0, \ldots, r\}$. (Here, we used the fact that in direction of $m^\perp$ higher derivatives of $p_{h,1}$ and $p_{h,2}$ coincide, even though s_h is only in $C^r(D_1 \cup D_2)$.) Since $(\beta-\mu)+(k+1+\mu) = \beta + k + 1 \le r + k + 1$, $\beta - \mu \ge 0$ and $k + 1 + \mu \ge k + 1$, $\mu = 0, \ldots, \beta$, $\beta = 0, \ldots, r$, we can apply (14) so that by (19) and (20),

$$|(f - p_{h,2})_{(m^\perp)^{k+1} l^\beta}(z)| \le (|\alpha_1| + |\alpha_2|)^\beta K_1 h^{q-k-\beta} \ , \tag{21}$$

for all $h \in (0, 1]$ and $\beta \in \{0, \ldots, r\}$. Thus, we get (17) with $K_4 = 2^r K_1$.
Now, we show that there exists a constant $K_5 > 0$ such that for all $h \in (0, 1]$ and $\beta \in \{0, \ldots, r\}$,

$$|(f - p_{h,2})_{(m^\perp)^{k+1-\mu} l^{\mu+\beta}}(z)| \le K_5 h^{q-k-\beta}, \qquad \mu = 1, \ldots, k + 1 \ . \tag{22}$$

Since $m^\perp = \alpha_3 l^\perp + \alpha_4 l$, where $\alpha_3 = \langle m^\perp, l^\perp\rangle$ and $\alpha_4 = \langle m^\perp, l\rangle$, we obtain from (18),

$$\begin{aligned}(f - p_{h,2})&_{(m^\perp)^{k+1-\mu} l^{\mu+\beta}}(z) = \\ &\sum_{\tau=0}^{k+1-\mu} \binom{k+1-\mu}{\tau} \alpha_3^{k+1-\mu-\tau} \alpha_4^{\tau} (f - p_{h,2})_{(l^\perp)^{k+1-\mu-\tau} l^{\mu+\beta+\tau}}(z) \ ,\end{aligned} \tag{23}$$

for all $\beta \in \{0, \ldots, r\}$ and $\mu \in \{1, \ldots, k+1\}$. Since $\mu+\beta+\tau \in \{0, \ldots, q-k-1+\mu+\tau\}$ and $k + 1 - \mu - \tau \in \{0, \ldots, k\}$, $\tau = 0, \ldots, k + 1 - \mu$, $\mu = 1, \ldots, k + 1$, we obtain by (15) and (23),

$$|(f - p_{h,2})_{(m^\perp)^{k+1-\mu} l^{\mu+\beta}}(z)| \le (|\alpha_3| + |\alpha_4|)^{k+1-\mu} K_2 h^{q-k-\beta} \ ,$$

for all $h \in (0, 1]$, $\beta \in \{0, \ldots, r\}$ and $\mu \in \{1, \ldots, k + 1\}$. Therefore, (22) holds with an appropriate constant $K_5 > 0$.

Now, we prove (16). Since $l^\perp = \alpha_5 m^\perp + \alpha_6 l$, where $\alpha_5 = \langle m^\perp, l^\perp\rangle^{-1}$ and $\alpha_6 = -\langle m^\perp, l\rangle\alpha_5$, and by (18), (17) and (22),

$$\begin{aligned} |(f-p_{h,2})_{(l^\perp)^{k+1}l^\beta}(z)| &\leq |\alpha_5^{k+1}(f-p_{h,2})_{(m^\perp)^{k+1}l^\beta}(z)| \\ &+ \sum_{\mu=1}^{k+1}\binom{k+1}{\mu}|\alpha_5^{k+1-\mu}\alpha_6^\mu(f-p_{h,2})_{(m^\perp)^{k+1-\mu}l^{\beta+\mu}}(z)| \\ &\leq (|\alpha_5|^{k+1}K_4 + (|\alpha_5|+|\alpha_6|)^{k+1}K_5)h^{q-k-\beta}\ , \end{aligned}$$

for all $h \in (0,1]$ and $\beta \in \{0,\dots,r\}$. By denoting $K_3 = |\alpha_5|^{k+1}K_4 + (|\alpha_5|+|\alpha_6|)^{k+1}K_5$, we obtain (16). This proves the lemma. ■

By using Lemma 3, Lemma 4 and Lemma 7, we now prove Theorem 2.

Proof of Theorem 2. Let a partition Δ^1 of R be given. We note that the partition depends on h. But for simplicity in the following we omit h. Let $s_f \in S_q^r(\Delta^1)$, $q \geq 3.5r+1$, be the unique interpolating spline of f.
The method of proof is to show that for each triangle $T_{i,j}^{(k)}$ the polynomial $p_{i,j}^{(k)} = s_f|_{T_{i,j}^{(k)}} \in \tilde{\Pi}_q$ weakly interpolates f on $T_{i,j}^{(k)}$, $i=1,\dots,n_1$, $j=1,\dots,n_2$, $k=1,2$. For doing this, we argue locally by considering a fixed number of triangles around each subtriangle T of the partition. Therefore, for all triangles, we obtain inequality (2) with the same constant $\tilde{K} > 0$. Then Theorem 2 follows from Lemma 3. Thus, we have to show :

Claim 1. For each triangle T of the partition Δ^1, the polynomial $p = s_f|_T \in \tilde{\Pi}_q$ weakly interpolates f on T.

We denote by $d = (d_1, d_2)$ the unit vector in the direction of the diagonal from $z_{i-1,j-1}$ to $z_{i,j}$. First, we show :

Claim 2. For all $\alpha \in \{0,\dots,r\}$ the polynomial $(p_{i,j}^{(1)})_{(d^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates $f_{(d^\perp)^\alpha}$ on $[z_{i-1,j-1}, z_{i,j}]$, $i=1,\dots,n_1$, $j=1,\dots,n_2$.

We consider four cases to prove Claim 2.
Case 1. Let $i \in \{2,\dots,n_1\}$ and $j \in \{1,\dots,n_2-1\}$ be given. We set $u = z_{i-1,j-1}$, $w = z_{i,j}$ and denote by $\hat{z}$ the midpoint of the diagonal of $R_{i,j}$ (see Figure 5).

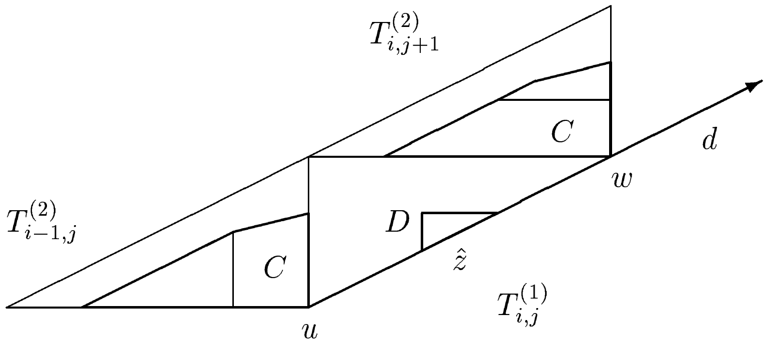

Figure 5.

We proceed by induction on α. It follows from the interpolation condition C on the triangles $T^{(2)}_{i-1,j}$ and $T^{(2)}_{i,j+1}$ at the points u and w, respectively, and the C^r property of s_f, that for all $\beta \in \{0, \dots, r\}$,

$$(f - p^{(1)}_{i,j})_{d^\beta}(u) = 0 \quad \text{and} \quad (f - p^{(1)}_{i,j})_{d^\beta}(w) = 0 \; . \tag{24}$$

Therefore, by the interpolation condition D on the triangle $T^{(1)}_{i,j}$ $((f - p^{(1)}_{i,j})_{d^\beta}(\hat{z}) = 0,\ \beta = 0, \dots, q - 2r - 2)$, and (24), we obtain that Claim 2 holds for $\alpha = 0$. We now assume that Claim 2 holds for all $\alpha \in \{0, \dots, k\},\ k \leq r - 1$ and show that the same is true for $\alpha = k + 1$.
To this end we will apply Lemma 7 to $f - s_f$ twice : at the point $z = u$ with $m = x$, $m^\perp = y$ and $l = d$, as well as at the point $z = w$ with $m = y$, $m^\perp = x$ and $l = -d$. By induction hypothesis and Lemma 4 (see remarks after this lemma), it follows, that there exists a constant $K_2 > 0$ such that for all $h \in (0, 1]$,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^\alpha d^\beta}(z)| \leq K_2 h^{q+1-\alpha-\beta}, \qquad \beta = 0, \dots, q - \alpha,\ \alpha = 0, \dots, k \; , \tag{25}$$

at any point $z \in [u, w]$. Hence, condition (15) holds true in both cases $z = u$ and $z = w$. Thus we have to check condition (14). First, we consider the case $z = u$. Condition (14) now has the form

$$|(f - p^{(2)}_{i-1,j})_{x^\alpha y^\beta}(u)| \leq K_1 h^{q+1-\alpha-\beta}, \qquad \alpha + \beta \leq r + k + 1,\ \alpha \geq 0,\ \beta \geq k + 1 \; , \tag{26}$$

so that (α, β) are all the integer points in the triangle of Figure 6.

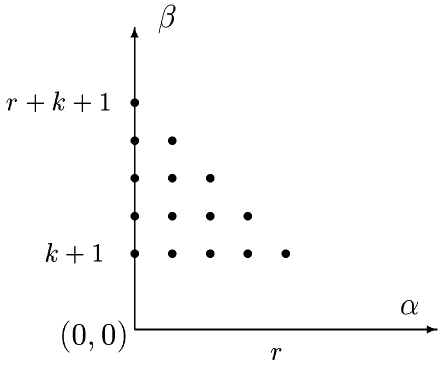

Figure 6.

On the other side, by the interpolation condition C on the triangle $T^{(2)}_{i-1,j}$, we have that

$$(f - p^{(2)}_{i-1,j})_{x^\alpha y^\beta}(u) = 0 \ , \tag{27}$$

for any (α, β) in the quadrangular domain shown in Figure 3. Therefore, the interpolation condition implies (26) if and only if the triangle lies inside the quadrangle (see Figure 7). It is easy to see that this is true for all $k \leq r-1$ while $q \geq 3.5r+1$.

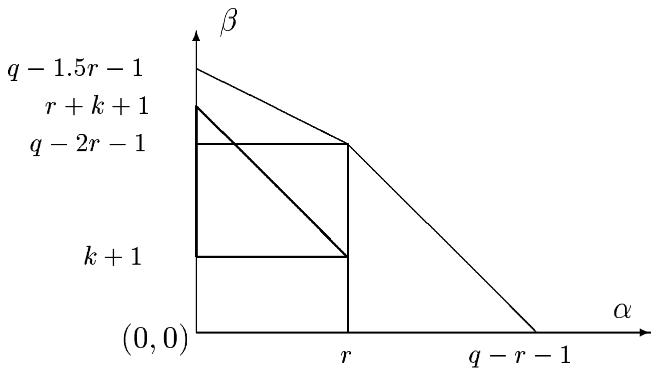

Figure 7.

Thus, by Lemma 7 there exists a constant $K_3 > 0$ such that for all $h \in (0,1]$ and $\beta \in \{0, \ldots, r\}$,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^{k+1} d^\beta}(u)| \leq K_3 h^{q-k-\beta} \ . \tag{28}$$

In the case $z = w$ condition (14) has the following form

$$(f - p^{(2)}_{i,j+1})_{y^\alpha x^\beta}(w) = 0, \qquad \alpha + \beta \leq r+k+1, \ \alpha \geq 0, \ \beta \geq k+1 \ . \tag{29}$$

The arguments similar to the above show that (29) follows from the interpolation condition C on the triangle $T^{(2)}_{i,j+1}$ if the triangle in Figure 8 lies inside the quadrangle. This is true for all $k \le r-1$ since $q \ge 3r+1$.

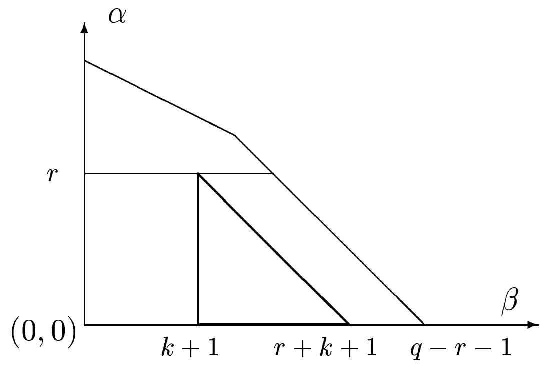

Figure 8.

By Lemma 7 there exists a constant $\tilde{K}_3 > 0$ such that for all $h \in (0,1]$ and $\beta \in \{0,\dots,r\}$,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^{k+1}d^\beta}(w)| = |(f - p^{(1)}_{i,j})_{(d^\perp)^{k+1}(-d)^\beta}(w)| \le \tilde{K}_3 h^{q-k-\beta} \; . \tag{30}$$

Now, it follows from (28) and (30) and by condition D,

$$(f - p^{(1)}_{i,j})_{(d^\perp)^{k+1}d^\beta}(\hat{z}) = 0, \quad \beta = 0,\dots,q-2r-k-3 \; ,$$

that Claim 2 holds true for $\alpha = k+1$.

Case 2. Let $i = 1$ and $j \in \{1,\dots,n_2-1\}$ be given. We set $u = z_{0,j-1}$, $v = z_{0,j}$ and $w = z_{1,j}$. Then $[u,v]$ lies on the boundary of the partition. By the interpolation condition B at u, we get for all $\beta \in \{0,\dots,q-r-1\}$,

$$(f - p^{(1)}_{1,j})_{d^\beta}(u) = 0 \; . \tag{31}$$

By the interpolation condition at w and the C^r property of s_f, we obtain for all $\beta \in \{0,\dots,r\}$,

$$(f - p^{(1)}_{1,j})_{d^\beta}(w) = 0 \; . \tag{32}$$

By (31) and (32), it follows that Claim 2 holds for $\alpha = 0$. We now assume that Claim 2 holds for all $\alpha \in \{0,\dots,k\}$, $k \le r-1$ and show that the same is true for $k+1$. We get (30) in the same way as in Case 1. By interpolation condition B at u, we obtain

$$(f - p^{(1)}_{i,j})_{(d^\perp)^{k+1}d^\beta}(u) = 0, \quad \beta = 0,\dots,q-r-k-2 \; . \tag{33}$$

It follows from (30) that Claim 2 holds for $\alpha = k+1$.
Case 3. Let $i \in \{2, \ldots, n_1\}$ and $j = n_2$. This case can be treated analogously as Case 2.
Case 4. Let $i = 1$ and $j = n_2$. Since $p^{(1)}_{1,n_2} \in \tilde{\Pi}_q$ has $dim\ \tilde{\Pi}_q$ interpolation conditions on $T^{(1)}_{1,n_2}$, this case is trivial.
Now, we show :

Claim 3. For all $\alpha \in \{0, \ldots, q-2r-1\}$ the polynomial $(p^{(2)}_{i,j})_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates f_{y^α} on $[z_{i-1,j-1}, z_{i,j-1}],\ i = 1, \ldots, n_1,\ j = 1, \ldots, n_2$.

To prove Claim 3, we proceed by induction on α. We set $u = z_{i-1,j-1},\ v = z_{i,j}$ and $w = z_{i,j-1}$. It follows from Claim 2 and Lemma 4, that there exists a constant $\tilde{K}_1 > 0$ such that for all $h \in (0,1]$,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^\alpha d^\beta}(u)| \leq \tilde{K}_1 h^{q+1-\alpha-\beta}, \quad \alpha = 0, \ldots, r,\ \beta \geq 0\ . \tag{34}$$

We have $x = \alpha_1 d + \alpha_2 d^\perp$, where $\alpha_1 = d_1,\ \alpha_2 = -d_2$. Hence by the C^r property of s_f and (18), we obtain from (34) that for all $h \in (0,1]$ and $\beta \in \{0, \ldots, r\}$,

$$|(f - p^{(2)}_{i,j})_{x^\beta}(u)| \leq \tilde{K}_1(|\alpha_1| + |\alpha_2|)^\beta h^{q+1-\beta}\ . \tag{35}$$

Interpolation condition C at w implies

$$(f - p^{(2)}_{i,j})_{x^\beta}(w) = 0, \quad \beta = 0, \ldots, q-r-1\ . \tag{36}$$

Therefore, we get from (35) and (36) that $p^{(2)}_{i,j}$ weakly interpolates f on $[u,w]$ so that the statement holds for $\alpha = 0$. We now assume that Claim 3 holds for all $\alpha \in \{0, \ldots, k\},\ k \leq q-2r-2$, and show that the same is true for $k+1$. By induction hypothesis and Lemma 4 it follows, that there exists a constant $\tilde{K}_2 > 0$ such that for all $h \in (0,1]$,

$$|(f - p^{(2)}_{i,j})_{y^\alpha x^\beta}(u)| \leq \tilde{K}_2 h^{q+1-\alpha-\beta}, \quad \beta = 0, \ldots, q-\alpha,\ \alpha = 0, \ldots, k\ . \tag{37}$$

We now apply Lemma 7 at the point $z = u$ with $m^\perp = d$ and $l = x$. It follows from (34) and (37) that conditions (14) and (15) of Lemma 7 are satisfied. Therefore, there exists a constant $\hat{K}_3 > 0$ such that for all $h \in (0,1]$ and $\beta \in \{0, \ldots, r\}$,

$$|(f - p^{(2)}_{i,j})_{y^{k+1}x^\beta}(u)| \leq \hat{K}_3 h^{q-k-\beta}\ . \tag{38}$$

By the interpolation condition C (respectively B, if $j = n_2$ or $i = n_1$) at w, we get

$$(f - p^{(2)}_{i,j})_{y^{k+1}x^\beta}(w) = 0, \quad \beta = 0, \ldots, q-k-r-2\ . \tag{39}$$

Thus, it follows from (38) and (39) that Claim 3 holds for $k+1$.
We remark that, if $T^{(2)}_{i,j}$ is such that s_f has interpolation condition B at w (i.e., $j = n_2$

or $i = n_1$), then it can be shown by analogue arguments that Claim 3 holds also for $\alpha \in \{q-2r, \ldots, q-r-1\}$.
In the following, we show :

Claim 4. For each triangle $T^{(2)}_{i,j}$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$ the polynomial $p^{(2)}_{i,j} \in \tilde{\Pi}_q$ weakly interpolates f on $T^{(2)}_{i,j}$.

To prove Claim 4, we consider three cases.
Case 1. Let $i \in \{1, \ldots, n_1 - 1\}$ and $j \in \{1, \ldots, n_2 - 1\}$. We set $u = z_{i-1,j-1}$, $v = z_{i,j}$ and $w = z_{i,j-1}$ (see Figure 9.).
Because of the C^r property of s_f and the fact that higher derivatives of $f - p^{(2)}_{i+1,j+1}$ and $f - p^{(1)}_{i+1,j}$ (respectively, $f - p^{(1)}_{i+1,j}$ and $f - p^{(2)}_{i,j}$) in direction of x (respectively, y) coincide, we get by Claim 3 (since $r \leq q - 2r - 1$) and Lemma 4 that there exists a constant $K_4 > 0$ such that for all $h \in (0, 1]$ and $\alpha,\ \beta \in \{0, \ldots, r\}$,

$$|(f - p^{(2)}_{i,j})_{x^\alpha y^\beta}(v)| = |(f - p^{(1)}_{i+1,j})_{x^\alpha y^\beta}(v)| = |(f - p^{(2)}_{i+1,j+1})_{x^\alpha y^\beta}(v)| \leq K_4 h^{q+1-\alpha-\beta} \ . \quad (40)$$

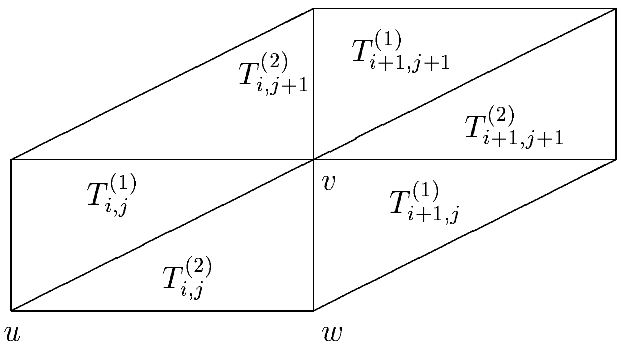

Figure 9.

In the following, we will show that for all $\alpha \in \{q-2r, \ldots, q-r-1\}$ the polynomial $(p^{(2)}_{i,j})_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates f_{y^α} on $[u, w]$. We prove this by induction on α. We begin with $\alpha = q - 2r$. It follows from (40) that for all $h \in (0, 1]$ and $\beta \in \{0, \ldots, r\}$,

$$|(f - p^{(2)}_{i,j})_{x^r y^\beta}(v)| \leq K_4 h^{q+1-r-\beta} \ . \quad (41)$$

By interpolation condition C at w, we get for all $\beta \in \{0, \ldots, q - 2r - 1\}$,

$$(f - p^{(2)}_{i,j})_{x^r y^\beta}(w) = 0 \ . \quad (42)$$

Therefore, by (41) and (42), we obtain that $(p_{i,j}^{(2)})_{x^r} \in \tilde{\Pi}_{q-r}$ weakly interpolates f_{x^r} on $[v,w]$. It follows from Lemma 4, that there exists a constant $K_5 > 0$ such that for all $h \in (0,1]$,

$$|(f - p_{i,j}^{(2)})_{y^{q-2r}x^r}(w)| \leq K_5 h^{r+1} \ . \tag{43}$$

Moreover, interpolation condition C at w also implies

$$(f - p_{i,j}^{(2)})_{y^{q-2r}x^\beta}(w) = 0, \quad \beta = 0, \dots, r-2 \ . \tag{44}$$

By Claim 3, $(p_{i,j}^{(2)})_{y^\alpha}$ weakly interpolates f_{y^α} on $[u,w]$, $\alpha = 0, \dots q-2r-1$. Hence, by Lemma 4, (37) holds true for $k = q-2r-1$. Therefore, arguing as in the proof of Claim 3, we obtain (38) with $k = q-2r-1$, i.e.,

$$|(f - p_{i,j}^{(2)})_{y^{q-2r}x^\beta}(u)| \leq \bar{K}_3 h^{2r+1-\beta}, \quad \beta = 0, \dots, r \ , \tag{45}$$

where $\bar{K}_3 > 0$. Since Birkhoff interpolation for $0 \leq t_0 = \dots = t_{r-1} < t_r = \dots = t_{2r}$, and $\gamma_j = j$, $j = 0, \dots, r-2$, $\gamma_{r-1} = r$, $\gamma_{r+j} = j$, $j = 0, \dots, r$, is easily seen to be well-posed for Π_{2r}, it follows from (43), (44) and (45) that $(p_{i,j}^{(2)})_{y^{q-2r}} \in \tilde{\Pi}_{2r}$ weakly interpolates $f_{y^{q-2r}}$ on $[u,w]$. This shows the case $\alpha = q-2r$.
Now, we assume that for $k \in \{q-2r, \dots, q-r-2\}$ the polynomial $(p_{i,j}^{(2)})_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates f_{y^α} on $[u,w]$ for all $\alpha \in \{q-2r, \dots, k\}$. Moreover, in the case $k \in \{q-2r, \dots, q-2r+[\frac{r}{2}]-1\}$, we assume that $(p_{i,j}^{(2)})_{x^{q-r-\alpha}} \in \tilde{\Pi}_{r+\alpha}$ weakly interpolates $f_{x^{q-r-\alpha}}$ on $[v,w]$ for all $\alpha \in \{q-2r, \dots, k\}$.
We have to show that $(p_{i,j}^{(2)})_{y^{k+1}} \in \tilde{\Pi}_{q-k-1}$ weakly interpolates $f_{y^{k+1}}$ on $[u,w]$. For doing this, we consider two cases. First, we treat the case $k \leq q-2r+[\frac{r}{2}]-1$.
It follows from (40) that for all $h \in (0,1]$ and $\beta \in \{0, \dots, r\}$,

$$|(f - p_{i,j}^{(2)})_{x^{q-r-k-1}y^\beta}(v)| \leq K_4 h^{r+k+2-\beta} \ . \tag{46}$$

By interpolation condition C at w, we get for all $\beta \in \{0, \dots, q-2r-1\}$,

$$(f - p_{i,j}^{(2)})_{x^{q-r-k-1}y^\beta}(w) = 0 \ . \tag{47}$$

¿From the induction hypothesis and Lemma 4, we obtain that there exists a constant $K_6 > 0$ such that for all $h \in (0,1]$ and $\beta \in \{q-2r, \dots, k\}$,

$$|(f - p_{i,j}^{(2)})_{x^{q-r-k-1}y^\beta}(w)| \leq K_6 h^{r+k+2-\beta} \ . \tag{48}$$

It follows from (46), (47) and (48) that $(p_{i,j}^{(2)})_{x^{q-r-k-1}} \in \tilde{\Pi}_{r+k+1}$ weakly interpolates $f_{x^{q-r-k-1}}$ on $[v,w]$. Hence, we obtain from Lemma 4 that there exists a constant $\tilde{K}_5 > 0$ such that for all $h \in (0,1]$,

$$|(f - p_{i,j}^{(2)})_{y^{k+1}x^{q-r-k-1}}(w)| \leq \tilde{K}_5 h^{r+1} \ . \tag{49}$$

The second part of the induction hypothesis says that $(p_{i,j}^{(2)})_{x^{q-r-\alpha}} \in \tilde{\Pi}_{r+\alpha}$ weakly interpolates $f_{x^{q-r-\alpha}}$ on $[v,w]$ for all $\alpha \in \{q-2r,\dots,k\}$. Therefore, it follows from Lemma 4 and (49) that there exists a constant $K_7 > 0$ such that for all $h \in (0,1]$ and $\beta \in \{q-r-k-1,\dots,r\}$,

$$|(f - p_{i,j}^{(2)})_{y^{k+1}x^{\beta}}(w)| \le K_7 h^{q-k-\beta} \ . \tag{50}$$

Moreover, interpolation condition C at w also implies

$$(f - p_{i,j}^{(2)})_{y^{k+1}x^{\beta}}(w) = 0, \qquad \beta = 0,\dots,2(q-k-2)-3r \ . \tag{51}$$

By the induction hypothesis and Claim 3, $(p_{i,j}^{(2)})_{y^{\alpha}}$ weakly interpolates $f_{y^{\alpha}}$ on $[u,w]$, for any $\alpha \in \{0,\dots,k\}$. Therefore, we can apply the same argumentation as in the proof of Claim 3 and obtain that (38) holds for the actual k, i.e.,

$$|(f - p_{i,j}^{(2)})_{y^{k+1}x^{\beta}}(u)| \le \check{K}_3 h^{q-k-\beta} \ , \qquad \beta = 0,\dots,r \ , \tag{52}$$

where $\check{K}_3 > 0$. It can be easily seen (cf. [7], Theorem 1.5) that Birkhoff interpolation for $0 \le t_0 = \dots = t_{q-k-r-2} < t_{q-k-r-1} = \dots = t_{q-k-1} \le 1$ and $\gamma_j = j$, $j = 0,\dots,2(q-k-2)-3r$, $\gamma_j = j-q+k+2r+2$, $j = 2(q-k-2)-3r+1,\dots,q-k-r-2$, $\gamma_{q-k-r-1+j} = j$, $j = 0,\dots,r$, is well-posed for Π_{q-k-1}. The hypotheses of Theorem 1.5 in [7] hold because the accompanying normal (algebraic) interpolation matrix satisfies the Polya condition and (trivially) contains no odd supported sequences. Therefore, it follows from (50), (51) and (52), that $(p_{i,j}^{(2)})_{y^{k+1}} \in \tilde{\Pi}_{q-k-1}$ weakly interpolates $f_{y^{k+1}}$ on $[u,w]$.
Now, we consider the case $k \in \{q-2r+[\frac{r}{2}],\dots,q-r-2\}$. Then it follows from the above consideration that $(p_{i,j}^{(2)})_{x^{\beta}} \in \tilde{\Pi}_{q-\beta}$ weakly interpolates $f_{x^{\beta}}$ on $[v,w]$ for all $\beta \in \{r-[\frac{r}{2}],\dots,r\}$. Hence, it follows from Lemma 4, that there exists a constant $K_8 > 0$ such that for all $h \in (0,1]$ and for all $\beta \in \{r-[\frac{r}{2}],\dots,r\}$, $\gamma \in \{0,\dots,q-\beta\}$,

$$|(f - p_{i,j}^{(2)})_{x^{\beta}y^{\gamma}}(w)| \le K_8 h^{q+1-\beta-\gamma} \ . \tag{53}$$

In particular, we get for all $h \in (0,1]$ and $\beta \in \{r-[\frac{r}{2}],\dots,q-k-[\frac{r}{2}]-2\}$,

$$|(f - p_{i,j}^{(2)})_{y^{k+1}x^{\beta}}(w)| \le K_8 h^{q-k-\beta} \ . \tag{54}$$

The same argumentation as above shows that (52) again holds. By [7], Theorem 1.5 the Birkhoff interpolation for $0 \le t_0 = \dots = t_{q-k-r-2} < t_{q-k-r-1} = \dots = t_{q-k-1} \le 1$ and $\gamma_j = r-[\frac{r}{2}]+j$, $j = 0,\dots,q-k-r-2$, $\gamma_{q-k-r-1+j} = j$, $j = 0,\dots,r$, is well-posed for Π_{q-k-1}. Therefore, it follows by (54) and (52) that $(p_{i,j}^{(2)})_{y^{k+1}} \in \tilde{\Pi}_{q-k-1}$ weakly interpolates $f_{y^{k+1}}$ on $[u,w]$.

Thus, we have shown that for all $\alpha \in \{q-2r, \ldots, q-r-1\}$ the polynomial $(p_{i,j}^{(2)})_{y^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates f_{y^α} on $[u, w]$.
By Claim 3 the same is also true for any $\alpha \in \{0, \ldots, q-2r-1\}$. An application of Lemma 4 shows that

$$\|(f - p_{i,j}^{(2)})_{y^\alpha x^\beta}\|_{[u,w]} \leq K_9 h^{q+1-\alpha-\beta} \ , \ \beta = 0, \ldots, q-\alpha, \ \alpha = 0, \ldots, q-r-1 \ ,$$

where $K_9 > 0$, particulary,

$$|(f - p_{i,j}^{(2)})_{y^\alpha x^\beta}(w)| \leq K_9 h^{q+1-\alpha-\beta} \ , \ \beta = 0, \ldots, q-\alpha, \ \alpha = 0, \ldots, q-r-1 \ . \tag{55}$$

In addition, we have by (40),

$$|(f - p_{i,j}^{(2)})_{y^\alpha x^\beta}(v)| \leq K_4 h^{q+1-\alpha-\beta}, \quad \beta = 0, \ldots, r-\alpha, \ \alpha = 0, \ldots, r \ . \tag{56}$$

It is easy to see that (55) and (56) together form a complete set of weak interpolation conditions for the triangle $T_{i,j}^{(2)}$. Therefore, the polynomial $p_{i,j}^{(2)} \in \tilde{\Pi}_q$ weakly interpolates f on $T_{i,j}^{(2)}$. This shows Claim 4 in Case 1.
Case 2. Let $i = n_1$ and $j \in \{1, \ldots, n_2 - 1\}$. We set $u = z_{n_1-1,j}, \ v = z_{n_1,j+1}$ and $w = z_{n_1,j}$.
By the same arguments as in the proof of Claim 3 (see remark at the end of the proof of Claim 3), we obtain by interpolation condition B at w that there exists a constant $K_{10} > 0$ such that for all $h \in (0, 1]$,

$$|(f - p_{n_1,j}^{(2)})_{y^\alpha x^\beta}(w)| \leq K_{10} h^{q+1-\alpha-\beta}, \quad \beta = 0, \ldots, q-\alpha, \ \alpha = 0, \ldots, q-r-1 \ . \tag{57}$$

It follows by interpolation condition B at v and the C^r property of s_f, that

$$(f - p_{n_1,j}^{(2)})_{y^\alpha x^\beta}(v) = (f - p_{n_1,j+1}^{(2)})_{y^\alpha x^\beta}(v) = 0, \quad \beta = 0, \ldots, r-\alpha, \ \alpha = 0, \ldots, r \ . \tag{58}$$

We now get from (57) and (58) that the polynomial $p_{i,j}^{(2)} \in \tilde{\Pi}_q$ weakly interpolates f on $T_{i,j}^{(2)}$. This shows Claim 4 in Case 2.
Case 3. Let $i \in \{1, \ldots, n_1\}$ and $j = n_2$. We set $u = z_{i,n_2-1}, \ v = z_{i+1,n_2}$ and $w = z_{i+1,n_2-1}$.
Now, we argue analogously as in Case 2 with the following difference. If $i > 1$, then we get (58) by using interpolation condition B at v concerning the triangle $T_{i,n_2}^{(1)}$. If $i = 1$, we get (58) by the fact that $p_{1,n_2}^{(1)} \in \tilde{\Pi}_q$ weakly interpolates f on the triangle $T_{1,n_2}^{(1)}$ (compare Claim 2, Case 4).
Now, we show the following :

Claim 5. For each triangle $T_{i,j}^{(1)}$, $i = 1, \ldots, n_1$, $j = 1, \ldots, n_2$ the polynomial $p_{i,j}^{(1)} \in \tilde{\Pi}_q$ weakly interpolates f on $T_{i,j}^{(1)}$.

To prove Claim 5, we consider three cases.
Case 1. Let $i \in \{2, \ldots, n_1\}$, $j \in \{1, \ldots, n_2 - 1\}$ be given. We choose u and w as in Case 1 of Claim 2 and $v = z_{i-1,j}$. Claim 4 shows that $p^{(2)}_{i-1,j} \in \tilde{\Pi}_q$ ($p^{(2)}_{i,j+1} \in \tilde{\Pi}_q$) weakly interpolates f on $T^{(2)}_{i-1,j}(T^{(2)}_{i,j+1})$. Thus, we get from Lemma 3 that there exist constants $K_{11} > 0$ and $K_{12} > 0$ such that for all $h \in (0, 1]$,

$$|(f - p^{(2)}_{i-1,j})_{x^\alpha y^\beta}(u)| \leq K_{11} h^{q+1-\alpha-\beta}, \quad \alpha + \beta \leq q,\ \alpha, \beta \geq 0 , \tag{59}$$

and

$$|(f - p^{(2)}_{i,j+1})_{y^\alpha x^\beta}(w)| \leq K_{12} h^{q+1-\alpha-\beta}, \quad \alpha + \beta \leq q,\ \alpha, \beta \geq 0 . \tag{60}$$

Using the same arguments as in the proof of Claim 2, Case 1, we obtain that $(p^{(1)}_{i,j})_{(d^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates $f_{(d^\perp)^\alpha}$ on $[u, w]$ for all $\alpha \in \{0, \ldots, q - r - 1\}$.
Therefore, by Lemma 4,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^\alpha d^\beta}(w)| \leq K_{13} h^{q+1-\alpha-\beta}, \quad \beta = 0, \ldots, q - \alpha,\ \alpha = 0, \ldots, q - r - 1 , \tag{61}$$

where $K_{13} > 0$. Because of the C^r property of s_f and Claim 4, we get by Lemma 3 that there exists a constant $K_{14} > 0$ such that for all $h \in (0, 1]$,

$$|(f - p^{(1)}_{i,j})_{(d^\perp)^\alpha d^\beta}(v)| \leq K_{14} h^{q+1-\alpha-\beta}, \quad \beta = 0, \ldots, r - \alpha,\ \alpha = 0, \ldots, r . \tag{62}$$

It follows from (61) and (62) that the polynomial $p^{(1)}_{i,j} \in \tilde{\Pi}_q$ weakly interpolates f on $T^{(1)}_{i,j}$.
Case 2. Let $i = 1$ and $j \in \{1, \ldots, n_2 - 1\}$ and u, v and w be choosen as in Case 2 of Claim 2. Then we use (60) and the same arguments as in Case 2 of Claim 2 to show that $(p^{(1)}_{1,j})_{(d^\perp)^\alpha} \in \tilde{\Pi}_{q-\alpha}$ weakly interpolates $f_{(d^\perp)^\alpha}$ on $[u, w]$ for all $\alpha \in \{0, \ldots, q - r - 1\}$. Moreover, we get (62) in the same way as in Case 1. This shows that the polynomial $p^{(1)}_{1,j} \in \tilde{\Pi}_q$ weakly interpolates f on $T^{(1)}_{1,j}$.
Case 3. Let $i \in \{1, \ldots, n_1\}$ and $j = n_2$. If $i > 1$, this case can be treated analogously to Case 2, with the difference that (59) is used instead of (60). If $i = 1$, Claim 5 is trivial.
Claim 1 now follows from Claim 4 and Claim 5. This proves the Theorem. ■

Remark 8. The results above also hold for splines defined on any simply connected subset of the rectangle R which is the union of the given subtriangles such that every pair of successive triangles has a common edge.

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