Approximation by Piecewise Constants on Convex Partitions

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Abstract

We show that the saturation order of piecewise constant approximation in L_p norm on convex partitions with N cells is $N^{-2/(d+1)}$, where d is the number of variables. This order is achieved for any $f \in W_p^2(\Omega)$ on a partition obtained by a simple algorithm involving an anisotropic subdivision of a uniform partition. This improves considerably the approximation order $N^{-1/d}$ achievable on isotropic partitions. In addition we show that the saturation order of piecewise linear approximation on convex partitions is $N^{-2/d}$, the same as on isotropic partitions.

1 Introduction

Let Ω be a bounded domain in $\mathbb{R}^d$, $d \geq 2$. Suppose that Δ is a *partition* of Ω into a finite number of subdomains $\omega \subset \Omega$ called *cells*, such that $\omega \cap \omega' = \emptyset$ if $\omega \neq \omega'$, and $\sum_{\omega\in\Delta} |\omega| = |\Omega|$, where $|\omega|$ denotes the Lebesgue measure (d-dimensional volume) of ω. A partition is said to be *convex* if each cell ω is a convex domain. We assume throughout the paper that Ω admits a convex partition. With a slight abuse of notation, we denote by $|D|$ the cardinality of a finite set D, so that $|\Delta|$ stands for the number of cells ω in Δ.

Given a function $f : \Omega \to \mathbb{R}$, we are interested in the error bounds for its approximation by piecewise polynomials in the space

$$S_n(\Delta) = \Big\{ \sum_{\omega\in\Delta} q_\omega \chi_\omega : \, q_\omega \in \Pi_n^d \Big\}, \qquad \chi_\omega(x) := \begin{cases} 1, & \text{if } x \in \omega, \\ 0, & \text{otherwise,} \end{cases}$$

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where Π_n^d, $n \geq 1$, is the space of polynomials of total degree $< n$ in d variables. The best approximation error is measured in the L_p-norm $\|\cdot\|_p := \|\cdot\|_{L_p(\Omega)}$,

$$E_n(f,\Delta)_p := \inf_{s\in S_n(\Delta)} \|f-s\|_p, \qquad 1 \leq p \leq \infty.$$

Clearly,

$$E_n(f,\Delta)_p = \begin{cases} \Big(\sum_{\omega\in\Delta} E_n(f)_{L_p(\omega)}^p\Big)^{1/p} & \text{if } \ p<\infty, \\ \max_{\omega\in\Delta} E_n(f)_{L_\infty(\omega)} & \text{if } \ p=\infty, \end{cases} \tag{1}$$

where

$$E_n(f)_{L_p(\omega)} := \inf_{q\in\Pi_n^d} \|f-q\|_{L_p(\omega)}$$

is the error of the best polynomial approximation of f on ω.

If ω is a bounded convex domain and $f_{|\omega}$ belongs to the Sobolev space $W_p^n(\omega)$, then the error $E_n(f)_{L_p(\omega)}$ is estimated as

$$E_n(f)_{L_p(\omega)} \leq C_{d,n} \operatorname{diam}^n(\omega)\, |f|_{W_p^n(\omega)}, \tag{2}$$

where $C_{d,n}$ denotes a positive constant depending only on d and n [3], and

$$|f|_{W_p^n(\omega)} := \sum_{|\alpha|=n} \Big\|\frac{\partial^n f}{\partial x^\alpha}\Big\|_{L_p(\omega)}, \qquad |\alpha| := \alpha_1+\cdots+\alpha_d \ \text{ for } \ \alpha\in\mathbb{Z}_+^d.$$

Note that

$$\|f-f_\omega\|_{L_p(\omega)} \leq 2E_1(f)_{L_p(\omega)}, \qquad f_\omega := |\omega|^{-1}\int_\omega f(x)\,dx,$$

see for example [2], and hence (2) implies that the Poincaré inequality

$$\|f-f_\omega\|_{L_p(\omega)} \leq \rho_d \operatorname{diam}(\omega)\|\nabla f\|_{L_p(\omega)}, \qquad f\in W_p^1(\omega), \tag{3}$$

holds with a constant ρ_d depending only on d when ω is bounded and convex, where

$$\|\nabla f\|_{L_p(\omega)} := \Big\|\Big(\sum_{k=1}^d \Big|\frac{\partial f}{\partial x_k}\Big|^2\Big)^{1/2}\Big\|_{L_p(\omega)}.$$

Indeed, it is easy to check that $\|\nabla f\|_{L_p(\omega)}$ is equivalent to the Sobolev seminorm $|f|_{W_p^1(\omega)}$, as

$$\|\nabla f\|_{L_p(\omega)} \leq |f|_{W_p^1(\omega)} \leq d^{\max\{\frac{1}{2},1-\frac{1}{p}\}}\|\nabla f\|_{L_p(\omega)}, \quad 1\leq p\leq\infty.$$

We prefer to use $\|\nabla f\|_{L_p(\omega)}$ in (3) because this seminorm is invariant under orthogonal transformations of the coordinate system, which simplifies some calculations below. It is important for the proof of Theorem 1 that ρ_d does not depend on the geometry of the domain.

It follows from (2) that for any convex partition Δ,

$$E_n(f,\Delta)_p \le C_{d,n}\,\mathrm{diam}^n(\Delta)\,|f|_{W_p^n(\Omega)}, \quad \mathrm{diam}(\Delta) := \max_{\omega\in\Delta} \mathrm{diam}(\omega).$$

Obviously, $\mathrm{diam}(\Delta) \ge C|\Delta|^{-1/d}$, where C depends only on $|\Omega|$ and d. Hence, in terms of $|\Delta|$, the approximation order that can be obtained from the last estimate is not better than

$$E_n(f,\Delta)_p = \mathcal{O}(|\Delta|^{-n/d}). \tag{4}$$

This order is achieved for example for $\Omega = (0,1)^d$ on convex partitions Δ_m, $m = 1, 2, \ldots$, defined by splitting the cube $(0,1)^d$ uniformly into $|\Delta_m| = m^d$ equal subcubes of edge length $1/m$.

Asymptotically optimal bounds for the L_p-error $e_n(f,\Delta)_p$ of the interpolation by piecewise polynomials of degree $< n$ on anisotropic triangulations of a polygonal domain in $\mathbb{R}^2$ have been studied in [1, 5]. There, for $n \ge 2$, an exact constant C_n is found such that $\liminf_{|\Delta_N|\to\infty} |\Delta_N|^{n/2} e_n(f,\Delta_N)_p \ge C_n$ as soon as the sequence of triangulations $\{\Delta_N\}$ satisfies $\mathrm{diam}(\Delta_N) = \mathcal{O}(|\Delta_N|^{-1/2})$. Moreover, a sequence $\{\Delta_N^*\}$ with this property exists such that $\limsup_{|\Delta_N|\to\infty} |\Delta_N^*|^{n/2} e_n(f,\Delta_N^*)_p \le C_n$.

In [2, Theorem 2] we have shown that assuming higher smoothness of f does not help to improve the order $E_1(f,\Delta_N)_\infty = \mathcal{O}(|\Delta_N|^{-1/d})$ if the sequence of partitions $\{\Delta_N\}$ is *isotropic*, that is there is a constant $c > 0$ such that $\mathrm{diam}(\omega) \le c\rho(\omega)$ for all $\omega \in \bigcup_N \Delta_N$, where $\rho(\omega)$ is the maximum diameter of d-dimensional balls contained in ω. More precisely, if $E_1(f,\Delta_N)_\infty = o(|\Delta_N|^{-1/d})$, $N \to \infty$, for a function $f \in C^1(\Omega)$ and some isotropic sequence of partitions $\{\Delta_N\}$ with $\lim\limits_{N\to\infty} \mathrm{diam}(\Delta_N) = 0$, then f is a constant. Thus, $|\Delta|^{-1/d}$ is the *saturation order* of the piecewise constant approximation on isotropic partitions.

In this paper we show that the order of approximation by piecewise constants can be improved to $E_1(f,\Delta)_p = \mathcal{O}(|\Delta|^{-2/(d+1)})$ on suitable anisotropic convex partitions obtained by a simple algorithm if $f \in W_p^2(\Omega)$, $\Omega = (0,1)^d$ (Algorithm 1 and Theorem 1). Moreover, according to Theorem 2, $|\Delta|^{-2/(d+1)}$ is the saturation order of piecewise constant approximation in L_∞-norm on convex partitions as it cannot be further improved for any $f \in C^2(\Omega)$ whose Hessian is positive definite at some point. Finally, Theorem 3 shows that the saturation order of piecewise linear approximations on convex partitions is $|\Delta|^{-2/d}$, that is the same as on isotropic partitions.

In the bivariate case the saturation order $N^{-2/3}$ has been shown by a different method in [4] for suitable sequences of partitions Δ_N of $(0,1)^2$ into polygons with cell boundaries consisting of totally $\mathcal{O}(N)$ straight line segments.

2 Optimal piecewise constant approximation on convex partitions

In this section we provide a simple algorithm that generates piecewise constant approximations with the approximation order $|\Delta|^{-2/(d+1)}$ on convex polyhedral partitions with totally $\mathcal{O}(|\Delta|)$ facets. For the sake of simplicity we only consider $\Omega = (0,1)^d$.

Algorithm 1. *Assume $f \in W_1^1(\Omega)$, $\Omega = (0,1)^d$. Split Ω into $N_1 = m^d$ cubes $\omega_1, \ldots, \omega_{N_1}$ of edge length $h = 1/m$. Then split each ω_i into N_2 slices ω_{ij}, $j = 1, \ldots, N_2$, by equidistant hyperplanes orthogonal to the average gradient $g_i := |\omega_i|^{-1} \int_{\omega_i} \nabla f(x)\,dx$ on ω_i. Set $\Delta = \{\omega_{ij} : i = 1, \ldots, N_1,\ j = 1, \ldots, N_2\}$, and define the piecewise constant approximation $s_\Delta(f)$ by*

$$s_\Delta(f) := \sum_{\omega \in \Delta} f_\omega \chi_\omega, \qquad f_\omega := |\omega|^{-1} \int_\omega f(x)\,dx. \tag{5}$$

Clearly, $|\Delta| = N_1 N_2$ and each ω_{ij} is a convex polyhedron with at most $2(d+1)$ facets.

This algorithm is illustrated in Fig. 1.

Theorem 1. *Assume that $f \in W_p^2(\Omega)$, $\Omega = (0,1)^d$, for some $1 \le p \le \infty$. For any $m = 1, 2, \ldots$, generate the partition Δ_m by using Algorithm 1 with $N_1 = m^d$ and $N_2 = m$. Then*

$$\|f - s_{\Delta_m}(f)\|_p \le C_d |\Delta_m|^{-2/(d+1)} (|f|_{W_p^1(\Omega)} + |f|_{W_p^2(\Omega)}), \tag{6}$$

where C_d is a constant depending only on d.

Proof. We only consider the case $p < \infty$ as $p = \infty$ can be derived by obvious modifications of the arguments given here. Note that a different proof in the case $p = \infty$ can be found in [2]. By construction,

$$\|f - s_{\Delta_m}(f)\|_p^p = \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \|f - f_{\omega_{ij}}\|_{L_p(\omega_{ij})}^p.$$

For a fixed i, let $\{\sigma_1, \ldots, \sigma_d\}$ be an orthonormal basis of $\mathbb{R}^d$ such that $\sigma_d = \|g_i\|^{-1} g_i$ if $g_i \ne 0$, and let $\varphi : \mathbb{R}^d \to \mathbb{R}^d$ be the linear mapping defined by

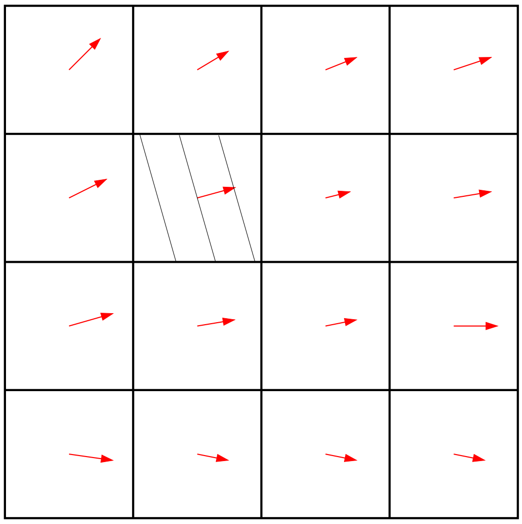

Figure 1: Algorithm 1 ($d = 2$, $N_2 = m = 4$). Average gradients g_i on the squares ω_i are depicted as arrows. The cells ω_{ij}, $j = 1, \dots, 4$, are shown only for one square.

the matrix $\mathrm{diag}(1, \dots, 1, N_2)$ with respect to the basis $\{\sigma_1, \dots, \sigma_d\}$. We set $\tilde{\omega}_{ij} = \varphi(\omega_{ij})$, $\tilde{f} = f \circ \varphi^{-1}$. Then $|\tilde{\omega}_{ij}| = N_2 |\omega_{ij}|$, $\mathrm{diam}(\tilde{\omega}_{ij}) \le d/m$, and

$$\|f - f_{\omega_{ij}}\|^p_{L_p(\omega_{ij})} = N_2^{-1} \|\tilde{f} - f_{\omega_{ij}}\|^p_{L_p(\tilde{\omega}_{ij})}.$$

Since $f_{\omega_{ij}} = \tilde{f}_{\tilde{\omega} ij}$ and $\tilde{\omega}_{ij}$ is bounded and convex, (3) shows that

$$\|\tilde{f} - f_{\omega_{ij}}\|_{L_p(\tilde{\omega}_{ij})} \le \rho_d \,\mathrm{diam}(\tilde{\omega}_{ij}) \|\nabla \tilde{f}\|_{L_p(\tilde{\omega}_{ij})},$$

where ρ_d depends only on d. We have

$$\begin{aligned} \|\nabla \tilde{f}\|^p_{L_p(\tilde{\omega}_{ij})} &= \Big\| \Big(\sum_{k=1}^{d} |D_{\sigma_k} \tilde{f}|^2 \Big)^{1/2} \Big\|^p_{L_p(\tilde{\omega}_{ij})} \\ &= N_2 \Big\| \Big(N_2^{-2} |D_{\sigma_d} f|^2 + \sum_{k=1}^{d-1} |D_{\sigma_k} f|^2 \Big)^{1/2} \Big\|^p_{L_p(\omega_{ij})} \\ &\le N_2^{1-p} \|D_{\sigma_d} f\|^p_{L_p(\omega_{ij})} + N_2 \sum_{k=1}^{d-1} \|D_{\sigma_k} f\|^p_{L_p(\omega_{ij})}, \end{aligned}$$

where $D_{\sigma_k} f = \nabla f^T \sigma_k$ denote the directional derivatives of f. Since

$$\int_{\omega_i} D_{\sigma_k} f(x)\, dx = 0, \qquad k = 1, \dots, d-1,$$

Poincaré inequality (3) also implies

$$\|D_{\sigma_k} f\|_{L_p(\omega_i)} \le \rho_d \operatorname{diam}(\omega_i) \|\nabla D_{\sigma_k} f\|_{L_p(\omega_i)}, \qquad k = 1, \dots, d-1.$$

Hence

$$\sum_{j=1}^{N_2} \sum_{k=1}^{d-1} \|D_{\sigma_k} f\|^p_{L_p(\omega_{ij})} \le d\Big(\frac{\sqrt{d}\rho_d}{m}\Big)^p |f|^p_{W^2_p(\omega_i)}.$$

By combining the above estimates we obtain

$$\begin{aligned}
\|f - s_{\Delta_m}(f)\|_p^p &\le \Big(\frac{d\rho_d}{m}\Big)^p N_2^{-1} \sum_{i=1}^{N_1} \sum_{j=1}^{N_2} \|\nabla \tilde{f}\|^p_{L_p(\tilde{\omega}_{ij})} \\
&\le \Big(\frac{d\rho_d}{m}\Big)^p \sum_{i=1}^{N_1} \Big[d\Big(\frac{\sqrt{d}\rho_d}{m}\Big)^p |f|^p_{W^2_p(\omega_i)} + N_2^{-p} \sum_{j=1}^{N_2} \|D_{\sigma_d} f\|^p_{L_p(\omega_{ij})} \Big] \\
&\le \Big(\frac{d\rho_d}{m}\Big)^p d\Big(\frac{\sqrt{d}\rho_d}{m}\Big)^p |f|^p_{W^2_p(\Omega)} + \Big(\frac{d\rho_d}{mN_2}\Big)^p |f|^p_{W^1_p(\Omega)}.
\end{aligned}$$

Since $N_1 = m^d, N_2 = m$, we have $|\Delta| = m^{d+1}$, and (6) follows with $C_d = d^{5/2}\rho_d^2$. □

3 Saturation Orders

The main result of this section is the following theorem which, together with Theorem 1 shows that the saturation order of piecewise constant approximation on convex partitions is $|\Delta|^{-2/(d+1)}$.

Theorem 2. *Assume that $f \in C^2(\Omega)$ and the Hessian of f is positive definite at a point $\hat{x} \in \Omega$. Then there is a constant $C_{f,d}$ depending only on f and d such that for any convex partition Δ of Ω,*

$$E_1(f, \Delta)_\infty \ge C_{f,d} |\Delta|^{-2/(d+1)}.$$

The proof of Theorem 2 will be given at the end of the section.

It turns out that *piecewise linear* approximations on convex partitions have the saturation order $|\Delta|^{-2/d}$. Thus, in contrast to piecewise constants, there is no improvement of the order in comparison to isotropic partitions.

Theorem 3. *Assume that $f \in C^2(\Omega)$ and the Hessian of f is positive definite at a point $\hat{x} \in \Omega$. Then there is a constant $C_{f,d}$ depending only on f and d such that for any convex partition Δ of Ω,*

$$E_2(f, \Delta)_\infty \ge C_{f,d} |\Delta|^{-2/d}.$$

Proof. Since $f \in C^2(\Omega)$, there is $\delta > 0$ and a cube $Q \subset \Omega$ such that the smallest eigenvalue of the Hessian of f is at least δ everywhere in Q.

Assume that $\omega \in \Delta$ has nonempty intersection with Q, and let $x_1, x_2 \in \omega \cap Q$ be such that $\|x_1 - x_2\|_2 \geq \frac{1}{2}\operatorname{diam}(\omega \cap Q)$. Since the univariate function $g := f_{|[x_1,x_2]}$ is convex with second derivative at least δ everywhere in $[x_1, x_2]$, the error of its best L_∞-approximation by (univariate) linear polynomials is greater or equal $\frac{\delta}{16}\|x_1 - x_2\|_2^2$. Indeed, by parametrising g with $t \in [0,1]$ and assuming without loss of generality that $g(0) = g(1) = 0$, we have $g''(t) \geq \delta\|x_1 - x_2\|_2^2$ and $g(t) = \frac{t(t-1)}{2}\int_0^1 g''(\tau)M_t(\tau)\,d\tau \leq \frac{t(t-1)}{2}\delta\|x_1 - x_2\|_2^2$, where M_t is the Peano kernel of the second divided difference $[0,1,t]$. Since $g(\frac{1}{2}) \leq -\frac{\delta}{8}\|x_1 - x_2\|_2^2$, Chebyshev theorem implies the claim.

Hence,

$$E_2(f,\Delta)_\infty \geq E_2(f)_{L_\infty(\omega\cap Q)} \geq \frac{\delta}{64}\operatorname{diam}^2(\omega \cap Q).$$

It follows that

$$|Q| \leq \frac{\mu_d}{2^d}\sum_{\omega\cap Q\neq\emptyset}\operatorname{diam}^d(\omega \cap Q) \leq \mu_d|\Delta|\left(\frac{16}{\delta}\right)^{d/2}E_2(f,\Delta)_\infty^{d/2},$$

where μ_d denotes the volume of the d-dimensional ball of radius 1. Thus,

$$E_2(f,\Delta)_\infty \geq \frac{\delta|Q|^{2/d}}{16\mu_d^{2/d}}|\Delta|^{-2/d}.$$

$\square$

Proof of Theorem 2. We first choose $\delta > 0$ and a cube $Q \subset \Omega$ such that the smallest eigenvalue of the Hessian of f is at least δ everywhere in Q. Clearly, $\nabla f(\tilde{x}) \neq 0$ for some $\tilde{x} \in Q$. Since the gradient of f is continuous, there is a constant $\gamma > 0$ and a cube $\tilde{Q} \subset Q$ containing $\tilde{x}$ such that $D_\sigma f(x) \geq \gamma$ for all $x \in \tilde{Q}$, where $\sigma = \nabla f(\tilde{x})/\|\nabla f(\tilde{x})\|_2$. We assume without loss of generality that $\tilde{Q} = Q$.

The arguments in the proof of Theorem 3 lead to the estimate

$$E_1(f,\Delta)_\infty \geq E_2(f,\Delta)_\infty \geq \frac{\delta}{64}\operatorname{diam}^2(\omega \cap Q)$$

for any $\omega \in \Delta$ with nonempty intersection with Q.

Moreover, if $[x_1, x_2]$ is an interval in $\omega \cap Q$ parallel to σ, then $|f(x_2) - f(x_1)| \geq \gamma\|x_2 - x_1\|_2$, which implies that

$$E_1(f,\Delta)_\infty \geq \frac{\gamma}{2}\|x_2 - x_1\|_2.$$

Hence $\omega \cap Q$ is contained between two hyperplanes orthogonal to σ, with distance between them not exceeding $\frac{2}{\gamma}E_1(f,\Delta)_\infty$. The penultimate display shows that the intersection of $\omega \cap Q$ with any intermediate hyperplane is contained in a $(d-1)$-dimensional ball of radius $\left(\frac{64}{\delta}E_1(f,\Delta)_\infty\right)^{1/2}$. Therefore, we may estimate the volume of $\omega \cap Q$ as

$$|\omega \cap Q| \le \frac{2}{\gamma}E_1(f,\Delta)_\infty \cdot \mu_{d-1}\Big(\frac{64}{\delta}E_1(f,\Delta)_\infty\Big)^{(d-1)/2},$$

which implies

$$|Q| \le |\Delta|\frac{2\mu_{d-1}}{\gamma}\Big(\frac{64}{\delta}\Big)^{(d-1)/2}E_1(f,\Delta)_\infty^{(d+1)/2},$$

and Theorem 2 follows. □

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