Interpolation by Splines on Triangulations

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Abstract

We review recently developed methods of constructing Lagrange and Hermite interpolation sets for bivariate splines on triangulations of general type. Approximation order and numerical performance of our methods are also discussed.

1 Introduction

Let Δ be a regular triangulation of a simply connected polygonal domain Ω in $\mathbb{R}^2$. Given integer r and q with $q \geq r+1$, we denote by $S_q^r(\Delta) = \{s \in C^r(\Omega) : s|_T \in \Pi_q \text{ for all } T \in \Delta\}$ the space of **bivariate splines of degree** q **and smoothness** r (with respect to Δ). Here $\Pi_q = \operatorname{span}\{x^\alpha y^\beta : \alpha, \beta \geq 0,\ \alpha+\beta \leq q\}$ denotes the space of **bivariate polynomials of total degree** q. We investigate the following problem. Construct **Lagrange interpolation set** $\{z_1, \ldots, z_d\}$ in Ω, where $d = \dim S_q^r(\Delta)$, such that for each function $f \in C(T)$, a unique spline $s \in S_q^r(\Delta)$ exists that satisfies the **Lagrange interpolation conditions** $s(z_\nu) = f(z_\nu)$, $\nu = 1, \ldots, d$. If we consider not only function values of f but also partial derivatives, then we speak of **Hermite interpolation conditions**.

In the literature, point sets that admit unique Lagrange and Hermite interpolation by spaces $S_q^r(\Delta)$ of splines of degree q and smoothness r were constructed for crosscut partitions Δ, in particular for Δ^1 and Δ^2-partitions [1, 4, 15, 21, 22, 23, 27, 28]. Results on the approximation order of these interpolation methods were given in [4, 11, 15, 20, 21, 24, 27, 28]. A Hermite interpolation scheme for $S_q^1(\Delta)$, $q \geq 5$, where Δ is an arbitrary triangulation, can be obtained by using a nodal basis of this space constructed in [17] (see also [9]). For $q = 4$ it was shown in [2] that a spline in $S_4^1(\Delta)$ exists which coincides with a given function at the vertices of Δ. Under certain restrictions on the triangulation, analogous results were obtained in [5, 18] for function and gradient values at the vertices. (Note that the dimension of $S_4^1(\Delta)$ is about six times the number of vertices of Δ.)

In this paper we review several new methods of interpolation by bivariate splines on triangulations.

In Section 2 we describe an inductive method [10] for constructing Lagrange and Hermite interpolation points for $S_q^1(\Delta)$, $q \geq 5$, where Δ is an arbitrary triangulation. Here, in each step, one vertex is added to the subtriangulation considered before. For $q = 4$ this method works under certain assumptions on Δ or a slight modification of it.

Section 3 is devoted to an algorithm [12] for constructing point sets that admit unique Lagrange and Hermite interpolation by the space $S_3^1(\Delta)$ of splines of degree 3 defined on a general class of triangulations Δ. Note that for $S_3^1(\Delta)$ even the dimension of the space is not known for arbitrary triangulations, in contrast to the case $q \geq 4$, where dimension formulas are available (cf. [17] for $q \geq 5$ and [2] for $q = 4$). We consider triangulations Δ that consist of nested polygons whose vertices are connected by line segments. In particular, the dimension of $S_3^1(\Delta)$ is determined for triangulations of this type.

In Section 4 we describe an algorithm [25, 26], which, for given points in the plane, constructs a triangulation Δ and, subsequently, Lagrange and Hermite interpolation sets for $S_q^r(\Delta)$, with $r = 1, 2$. Moreover, this method is applied to given quadrangulations with diagonals.

In Section 5 we discuss a Hermite type interpolation scheme [13] for $S_q^r(\Delta)$, $q \geq 3r + 2$, which possesses optimal approximation order $\mathcal{O}(h^{q+1})$. Furthermore, the fundamental functions of the scheme form a stable (for the triangulations that do not contain near-degenerate edges) and locally linearly independent basis for a superspline subspace of $S_q^r(\Delta)$.

Finally, in Section 6 some numerical examples are presented.

2 Interpolation by C^1 Splines of Degree $q \geq 4$

The choice of interpolation points depends on the following properties of edges, vertices and subtriangulations of Δ.

Definition 2.1. *(i) An interior edge e with vertex v of the triangulation Δ is called* `degenerate` *at v if the edges with vertex v adjacent to e lie on a line. (ii) An interior vertex v of Δ is called* `singular` *if v is a vertex of exactly four edges and these edges lie on two lines. (iii) An interior vertex v of Δ on the boundary of a given subtriangulation Δ' of Δ is called* `semi-singular of type 1` *w.r.t. Δ' if exactly one edge with endpoint v is not contained in Δ' and this edge is degenerate at v. (iv) An interior vertex v of Δ on the boundary of a given subtriangulation Δ' of Δ is called* `semi-singular of type 2` *w.r.t. Δ' if exactly two edges with endpoint v are not contained in Δ' and these edges are degenerate at v. (v) A vertex v of Δ is called* `semi-singular` *w.r.t. Δ' if v satisfies (iii) or (iv).*

Definition 2.2. *We say that $\Delta' \subset \Delta$ is a* `tame subtriangulation` *if the following conditions* (T1)–(T3) *hold.*

(T1) *$\Omega_{\Delta'} := \bigcup_{T \in \Delta'} T$ is simply connected.*

(T2) *For any two triangles $T', T'' \in \Delta'$ there exists a sequence $\{T_1, \ldots, T_\mu\} \subset \Delta'$ such that T_i and T_{i+1} have a common edge, $i = 1, \ldots, \mu - 1$, $T_1 = T'$, $T_\mu = T''$.*

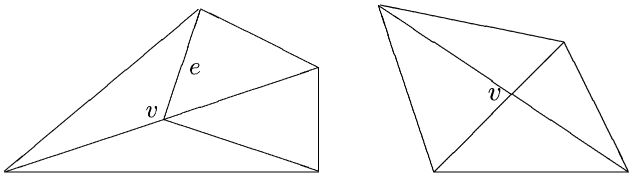

Fig. 2.1. Degenerate edge, respectively singular vertex.

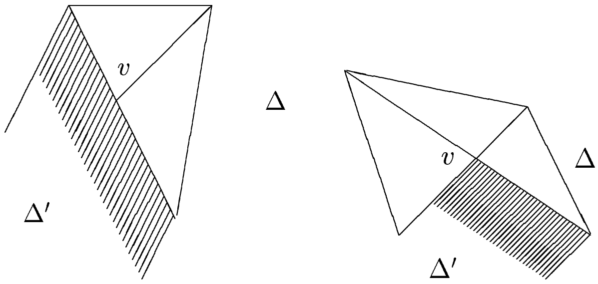

Fig. 2.2. Semi-singular vertex.

(T3) *If two vertices* $v_1, v_2 \in \Omega_{\Delta'}$ *are connected by an edge* e *of the triangulation* Δ, *then* $e \subset \Omega_{\Delta'}$.

Interpolation by C^1 Quartic Splines. We construct a chain of subsets Ω_i of Ω such that $\emptyset = \Omega_0 \subset \Omega_1 \subset \Omega_2 \subset \cdots \subset \Omega_m = \Omega$, and correspond to each Ω_i a set of points $\mathcal{L}_i \subset \Omega_i \setminus \Omega_{i-1}$, $i = 1, \ldots, m$.

For $i = 1$, we take $\Delta_1 = \{T_1\}$, where T_1 is an arbitrarily chosen "starting" triangle in Δ, and set $\Omega_1 := \Omega_{\Delta_1} = T_1$. We choose $\mathcal{L}_1$ to be an arbitrary set of 15 points lying on T_1 and admissible for Lagrange interpolation from Π_4. (For example, we choose five parallel line segments l_ν in T_1 and ν different points on each l_ν, $\nu = 1, 2, 3, 4, 5$.)

Proceeding by induction, we take $i \geq 2$ and suppose that Δ_{i-1} has already been defined and is a tame subtriangulation of Δ, with $\Omega_{i-1} := \Omega_{\Delta_{i-1}}$ being a proper subset of Ω. In order to construct Δ_i, we choose a vertex $v_i \in \Omega \setminus \Omega_{i-1}$ such that v_i is connected to vertices $v_{i,0}, v_{i,1}, \ldots, v_{i,\mu_i} \in \Omega_{i-1}$, where $\mu_i \geq 1$, and the subtriangulation $\Delta_i := \Delta_{i-1} \cup \{T_{i,1}, \ldots, T_{i,\mu_i}\}$, where $T_{i,j} := \langle v_i, v_{i,j-1}, v_{i,j} \rangle$, is tame. (Existence of at least one v_i with this property is shown in [10].) Thus, we set $\Omega_i := \Omega_{\Delta_i}$ (see Figure 2.3).

In order to describe $\mathcal{L}_i$, we need additional notation. Denote by $\hat{e}_{i,j}$ the edge attached to v_i and $v_{i,j}$, $j = 0, \ldots, \mu_i$, $i = 2, \ldots, m$ (see Figure 2.4). For each $i \in \{2, \ldots, m\}$ we define $J_i \subset \{0, \ldots, \mu_i\}$ as follows: 1) for $j \in \{1, \ldots, \mu_i - 1\}$, we have $j \in J_i$ if and only if $\hat{e}_{i,j}$ is nondegenerate at $v_{i,j}$; 2) for $j \in \{0, \mu_i\}$, we have

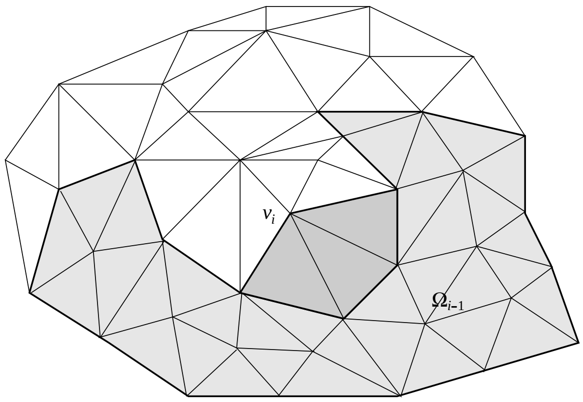

Fig. 2.3. Construction of subtriangulation Δ_i.

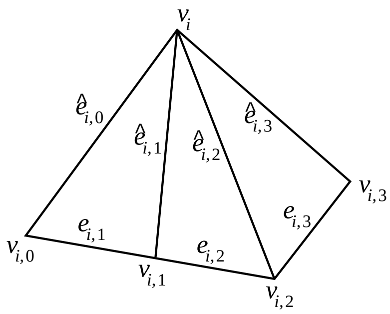

Fig. 2.4. $\Omega_i \setminus \Omega_{i-1}$.

$j \in J_i$ if and only if $v_{i,j}$ is semisingular w.r.t. Δ_i, and $\hat{e}_{i,j}$ is nondegenerate at $v_{i,j}$. Moreover, for every $i \in \{2, \ldots, m\}$ we set $\theta_i := 1$ if v_i is semisingular w.r.t. Δ_i but nonsingular, and $\theta_i := 0$ otherwise.

We consider three cases.

Case 1. Suppose that $\theta_i = 0$. Then $\mathcal{L}_i \subset \Omega_i \setminus \Omega_{i-1}$ consists of

- the vertex v_i,
- any point $w_{i,j}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus J_i$,
- two points w_i', w_i'' in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\}$, and
- any point z_i in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\}$.

Case 2. Suppose that $\theta_i = 1$ and there exists $j^* \in \{0, \mu_i\} \setminus J_i$, such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i. Then $\mathcal{L}_i \subset \Omega_i \setminus \Omega_{i-1}$ consists of

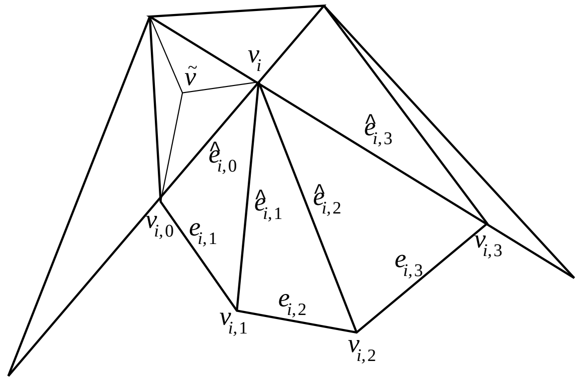

Fig. 2.5. Clough-Tocher split of a triangle in Case 3.

- the vertex v_i,
- any point $w_{i,j}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \dots, \mu_i\} \setminus (J_i \cup \{j^*\})$,
- two points w'_i, w''_i in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \dots, \mu_i\} \setminus \{j^*\}$, and
- any point z_i in the interior of a triangle $T_{i,j'''}$, for some $j''' \notin \{j^*, j^* + 1\}$.

Case 3. Suppose that $\theta_i = 1$ and $\hat{e}_{i,j}$ is degenerate at v_i for every $j \in \{0, \mu_i\} \setminus J_i$. Then we need to slightly modify the triangulation by performing a Clough-Tocher split of the triangle $\tilde{T}$ that lies outside Ω_i and shares the edge $\hat{e}_{i,0}$ with $T_{i,1}$. Therefore, we add a new vertex $\tilde{v}$ in the interior of $\tilde{T}$ and connect $\tilde{v}$ with three edges to each of the vertices of $\tilde{T}$ (see Figure 2.5). After this modification vertex v_i is no longer semisingular w.r.t. Δ_i, hence $\theta_i = 0$, and we choose $\mathcal{L}_i \subset \Omega_i \setminus \Omega_{i-1}$ according to the rule described in Case 1. Furthermore, we choose $v_{i+1} := \tilde{v}$. It is easy to see that Δ_{i+1} defined by adding to Δ_i the triangle with vertices v_i, $v_{i,0}$ and $\tilde{v}$, is a tame subtriangulation of Δ. Moreover, we have $\theta_{i+1} = 0$. Thus, we choose $\mathcal{L}_{i+1} \subset \Omega_{i+1} \setminus \Omega_i$ according to Case 1. We denote the resulting modified triangulation by Δ^*.

Theorem 2.3. [10] *The set of points $\mathcal{L} := \bigcup_{i=1}^m \mathcal{L}_i$ described above is a Lagrange interpolation set for $S_4^1(\Delta^*)$. In particular, $\Delta^* = \Delta$ if Case 3 does not occur.*

Remark 2.4. (i) We note that Case 3 is an exceptional case. Among other things, its occurrence requires that one vertex of Δ should be connected with five vertices lying on a line (if v_i is semisingular of type I) or two vertices of Δ should be connected with four vertices lying on a line (if v_i is semisingular of type II: see Figure 2.5). Therefore, no modification of Δ is needed if each vertex is connected

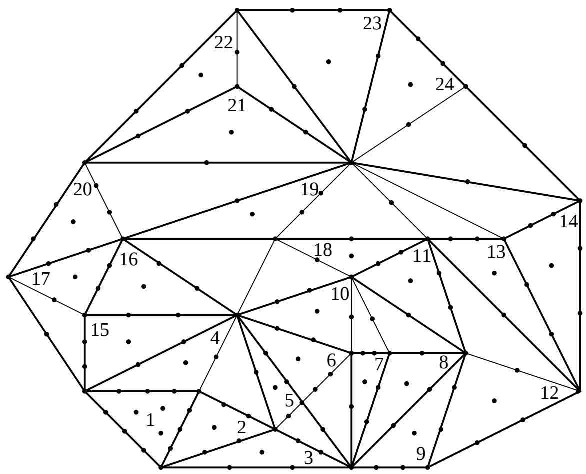

Fig. 2.6. Location of Lagrange interpolation points for $S_4^1(\Delta)$.

with at most three vertices lying on a line. In particular, this last property is satisfied for any triangulation obtained from an arbitrary convex quadrangulation by inserting one or two diagonals of each quadrilateral. (ii) We also note that our method works without modifying Δ if the total number of edges attached to v_i is odd. Then $\mathcal{L}_i$ is defined in Case 3 as in Case 1, with the point z_i being removed.

Remark 2.5. Lagrange interpolation of f at some points of the above scheme can be replaced by interpolation of appropriate first or second partial derivatives of f provided that such derivatives exist. Namely, interpolation of f at w_i', w_i'' can be replaced by the conditions $D_x s(v_i) = D_x f(v_i)$, $D_y s(v_i) = D_y f(v_i)$, interpolation of f at $w_{i,j}$ can be replaced by $D^2_{\hat{e}_{i,j}} s(v_i) = D^2_{\hat{e}_{i,j}} f(v_i)$, and interpolation of f at z_i can be replaced by $D_{\hat{e}_{i,j'''-1}} D_{\hat{e}_{i,j'''}} s(v_i) = D_{\hat{e}_{i,j'''-1}} D_{\hat{e}_{i,j'''}} f(v_i)$. (Here and below we use the notations $D_x f := \frac{\partial f}{\partial x}$, $D_y f := \frac{\partial f}{\partial y}$ and $D_\sigma f := \sigma_x D_x f + \sigma_y D_y f$, where $\sigma = (\sigma_x, \sigma_y)$ is a unit vector in the plane. For simplicity, we write $D_e f := D_\sigma f$, where σ is the unit vector parallel to the edge e.) Particularly, our Hermite interpolation scheme includes the function and gradient values at all vertices of the triangulation.

Remark 2.6. The computation of the interpolating spline $s \in S_4^1(\Delta)$ according to our scheme is easy to perform step by step, by constructing $s|_{\Omega_i \setminus \Omega_{i-1}}$ after $s|_{\Omega_{i-1} \setminus \Omega_{i-2}}$. This can always be done by solving small systems of linear equations. Moreover, for Δ^1 and Δ^2 triangulations our method leads to the interpolation schemes developed by Nürnberger [20] and Nürnberger & Walz [24], respectively. These schemes possess (nearly) optimal approximation order. (For certain classes of triangulations, quasi-interpolation methods for $S_4^1(\Delta)$ were developed in [5, 6].)

Interpolation by C^1 Splines of Degree $q \geq 5$. We construct a chain of subsets Ω_i of Ω as above and correspond to each Ω_i a set of points $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$, $i = 1, \ldots, m$, as follows. (In this case no modification of the given triangulation Δ is necessary.) Namely, we choose $\mathcal{L}_1^{(q)}$ to be an arbitrary set of $d_q := \binom{q+2}{2}$ points lying on T_1 and admissible for Lagrange interpolation from Π_q. In order to define $\mathcal{L}_i^{(q)}$ we consider two cases.

Case 1. Suppose that $\theta_i = 0$. Then $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of

- the vertex v_i,
- any $q-3$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-3)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus J_i$,
- any $q-4$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-4)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in J_i$,
- two points w_i', w_i'' in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\}$,
- any d_{q-4} distinct points $z_{i,j'''}^{(1)}, \ldots, z_{i,j'''}^{(d_{q-4})}$ lying in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\}$, and admissible for Lagrange interpolation from Π_{q-4}, and
- any d_{q-5} distinct points $z_{i,j}^{(1)}, \ldots, z_{i,j}^{(d_{q-5})}$ lying in the interior of $T_{i,j}$ and admissible for Lagrange interpolation from Π_{q-5}, for each $j \in \{1, \ldots, \mu_i\} \setminus \{j'''\}$.

Case 2. Suppose that $\theta_i = 1$. (Hence, there exists $j^* \in \{0, \mu_i\}$, such that $\hat{e}_{i,j^*}$ is nondegenerate at v_i.) Then $\mathcal{L}_i^{(q)} \subset \Omega_i \setminus \Omega_{i-1}$ consists of

- the vertex v_i,
- any $q-3$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-3)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in \{0, \ldots, \mu_i\} \setminus (J_i \cup \{j^*\})$,
- any $q-4$ distinct points $w_{i,j}^{(1)}, \ldots, w_{i,j}^{(q-4)}$ in the interior of the edge $\hat{e}_{i,j}$, for each $j \in J_i \setminus \{j^*\}$,
- any $q-\kappa$ distinct points $w_{i,j^*}^{(1)}, \ldots, w_{i,j^*}^{(q-\kappa)}$ in the interior of the edge $\hat{e}_{i,j^*}$, where $\kappa = 5$ if $j^* \in J_i$, and $\kappa = 4$ if $j^* \notin J_i$,
- two points w_i', w_i'' in the interiors of two noncollinear edges $\hat{e}_{i,j'}$ and $\hat{e}_{i,j''}$ respectively, for some $j', j'' \in \{0, \ldots, \mu_i\} \setminus \{j^*\}$,
- any d_{q-4} distinct points $z_{i,j'''}^{(1)}, \ldots, z_{i,j'''}^{(d_{q-4})}$ lying in the interior of a triangle $T_{i,j'''}$, for some $j''' \in \{1, \ldots, \mu_i\} \setminus \{j^*, j^*+1\}$, and admissible for Lagrange interpolation from Π_{q-4}, and

- any d_{q-5} distinct points $z_{i,j}^{(1)}, \dots, z_{i,j}^{(d_{q-5})}$ lying in the interior of $T_{i,j}$ and admissible for Lagrange interpolation from Π_{q-5}, for each $j \in \{1, \dots, \mu_i\} \setminus \{j'''\}$.

Theorem 2.7. [10] *The set of points* $\mathcal{L}^{(q)} := \bigcup_{i=1}^{m} \mathcal{L}_i^{(q)}$ *described above is a Lagrange interpolation set for* $S_q^1(\Delta)$, $q \geq 5$.

Remark 2.8. As in the case $q = 4$, our Lagrange interpolation scheme can be transformed into an appropriate Hermite interpolation scheme (cp. Remark 2.5). Moreover, Remark 2.6 about computation and approximation order of our interpolation method remains true in the case $q \geq 5$.

3 Interpolation by Cubic Splines

The Class of Triangulations. In this section we consider the following general type of triangulations Δ. The vertices of Δ are the vertices of closed simple polygons $P_0, P_1, \dots, P_k$ which are nested and one vertex inside P_0. This means that $\Omega_{\mu-1} \subset \Omega_\mu$, where Ω_μ is the closed (not necessarily convex) polyhedron with boundary $P_\mu, \mu = 0, \dots, k$, and Δ is a triangulation of $\Omega := \Omega_k$ (see Figure 3.1). To be more precise, we note that the vertices of P_μ are connected by line segments with the vertices of $P_{\mu+1}$, $\mu = 0, \dots, k-1$. On the other hand, for each closed simple polygon P_μ, there is no additional line segment connecting two vertices of P_μ, $\mu = 0, \dots, k$. In order to construct interpolation points for $S_3^1(\Delta)$, we assume that the triangulation Δ has the following properties:

(C1) Each vertex of P_μ is connected with at least two vertices of $P_{\mu+1}, \mu = 0, \dots, k-1$.

(C2) There exist vertices w_μ of $P_\mu, \mu = 0, \dots, k$, such that w_μ and $w_{\mu+1}$ are connected, and each vertex w_μ is connected with at least three vertices of $P_{\mu+1}, \mu = 0, \dots, k-1$.

Remark 3.1. (*i*) Since the polygons P_μ grow with increasing index μ, it is natural to assume that the number of vertices of $P_{\mu+1}$ is greater than the number of vertices of P_μ, $\mu = 0, \dots, k-1$. Then it is natural to connect the vertices of the polygons in such a way that the properties (C1) and (C2) are satisfied. (*ii*) Moreover, the properties (C1) and (C2) of Δ remain valid if Δ is `deformed`, i.e., the location of the vertices of Δ are changed but the connection of the vertices remain unchanged. (In other words, the graphs of the triangulation Δ and the deformed triangulation are the same.)

Decomposition of the Domain. In order to construct interpolation points, we decompose the domain Ω into finitely many sets $V_0 \subset V_1 \subset \ldots \subset V_m = \Omega$, where each set V_i, is the union of closed triangles of $\Delta, i = 0, \dots, m$. Let V_0 be an arbitrary

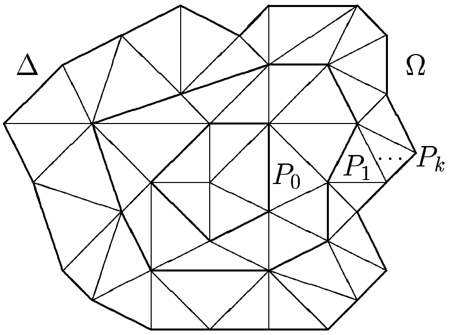

Fig. 3.1. Triangulation Δ (nested polygons).

closed triangle of Δ in Ω_0. We define the sets $V_1 \subset \ldots \subset V_m$ by induction according to the following `rule`: If V_{i-1} is defined, then we choose a vertex v_i of Δ with the following property: Let $T_{i,1}, \ldots, T_{i,n_i}(n_i \geq 1)$ be all triangles of Δ with vertex v_i having a common edge with V_{i-1}. (Since Δ satisfies property (C1), we have $n_i \leq 2$.) We set $V_i = V_{i-1} \cup \overline{T}_{i,1} \cup \ldots \cup \overline{T}_{i,n_i}$. (Note that we choose the vertex v_i in such a way that at least one such triangle exists.) The vertices $v_i, i = 1, \ldots, m$, are chosen as follows. After choosing V_0 to be an arbitrary closed triangle of Δ in Ω_0, we pass through the vertices of P_0 in clockwise order by applying the above rule. (It is clear that the choice of these vertices is unique.) Now, we assume that we have passed through the vertices of $P_{\mu-1}$. Then w.r.t. clockwise order, we choose the first vertex of P_μ greater than w_μ which is connected with at least two vertices of $P_{\mu-1}$. Then we pass through the vertices of P_μ in clockwise order until w_μ^- and pass through the vertices of P_μ in anticlockwise order until w_μ^+ by applying the above rule. (Here w_μ^+ denotes the vertex next to w_μ in clockwise order and w_μ^- denotes the vertex next to w_μ in anticlockwise order.) Finally, we choose the vertex w_μ. (It is clear that the choice of the vertices is unique.) In this way, we obtain the sets $V_0 \subset V_1 \subset \ldots \subset V_m = \Omega$.

Construction of Interpolation Sets. Now, we construct interpolation sets for $S_3^1(\Delta)$ inductively as follows. First, we choose interpolation points on V_0 and then on $V_i \setminus V_{i-1}, i = 1, \ldots, m$. In the first step, we choose 10 different points (respectively 10 Hermite interpolation conditions) on V_0 which admit unique Lagrange interpolation by the space Π_3. (For example, we may choose four parallel line segments l_ν in V_0 and ν different points on each $l_\nu, \nu = 1, 2, 3, 4$.)

Now, we assume that we have already chosen interpolation points on V_{i-1}. Then we choose interpolation points on $V_i \setminus V_{i-1}$ as follows. By the above decomposition of Ω, $V_i \setminus V_{i-1}$ is the union of consecutive triangles $T_{i,1}, \ldots, T_{i,n_i}$ with vertex v_i having common edges with V_{i-1}. We denote the consecutive endpoints of these edges by $v_{i,0}, v_{i,1}, \ldots, v_{i,n_i}$. Moreover, the edges $[v_{i,j}, v_i]$ are denoted by $e_{i,j}, j = 0, \ldots, n_i$ (see Figure 3.2).

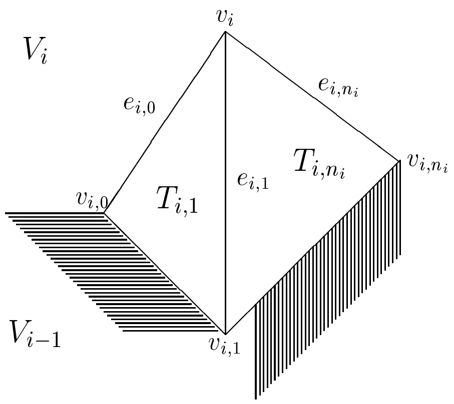

Fig. 3.2. The set $V_i \setminus V_{i-1}$.

The choice of interpolation points on $V_i \setminus V_{i-1}$ depends on the following properties of the subtriangulation $\Delta_i = \{T \in \Delta : T \subset V_i\}$ at the vertices $v_{i,0}, \ldots, v_{i,n_i}$: (i) $e_{i,j}$ in non-degenerate at $v_{i,j}$. (ii) $e_{i,j}$ is non-degenerate at $v_{i,j}$ and in addition, $v_{i,j}$ is semi-singular w.r.t. Δ_i.

For $j \in \{0, n_i\}$, we set $c_{i,j} = 1$ if (ii) holds; and $c_{i,j} = 0$ otherwise. For j with $0 < j < n_i$, we set $c_{i,j} = 1$ if (i) holds; and $c_{i,j} = 0$ otherwise. Moreover, we set $c_i = \sum_{j=0}^{n_i} c_{i,j}$ and note that $0 \leq c_i \leq 3$. For `Lagrange interpolation`, we choose the following points on $V_i \setminus V_{i-1}$: If $c_i = 3$, then no point is chosen. If $c_i = 2$, then we choose v_i. If $c_i = 1$, then we choose v_i and one further point on some edge $e_{i,j}$ with $c_{i,j} = 0$. If $c_i = 0$, then we choose v_i and two further points on two different edges. For `Hermite interpolation`, we require the following interpolation conditions for $s \in S_3^1(\Delta)$ at the vertex v_i: If $c_i = 3$, then no interpolation condition is required at v_i. If $c_i = 2$, then we require $s(v_i) = f(v_i)$. If $c_i = 1$, then we require $s(v_i) = f(v_i)$ and $D_{e_{i,j}} s(v_i) = D_{e_{i,j}} f(v_i)$, where $e_{i,j}$ is some edge with $c_{i,j} = 0$. If $c_i = 0$, then we require $s(v_i) = f(v_i)$, $D_x s(v_i) = D_x f(v_i)$ and $D_y s(v_i) = D_y f(v_i)$. By the above construction, we obtain a set of points for Lagrange interpolation respectively a set of Hermite interpolation conditions.

Theorem 3.2. [12] *If the triangulation Δ satisfies the properties* (C1) *and* (C2), *then a unique spline in $S_3^1(\Delta)$ exists which satisfies the above Lagrange (respectively Hermite) interpolation conditions. In particular, the total number of interpolation conditions is equal to the dimension of $S_3^1(\Delta)$.*

Corollary 3.3. *Let Δ be a deformed Δ^1-partition. Then a unique spline in $S_3^1(\Delta)$ exists which satisfies the Lagrange (respectively Hermite) interpolation conditions obtained by our method.*

We note that the basic principle of passing through the vertices of the nested polygons of Δ can also be applied to the space $S_q^1(\Delta)$, $q \geq 4$, in combination with the algorithm for constructing interpolation points in Section 2. Then, in contrast to Section 2, the choice of the vertices is unique as soon as the nested polygons, the starting triangle and the vertices w_μ have been identified.

4 Interpolation by Splines on Triangulations of Given Points

In this section, we construct a natural triangulation Δ for given points in the plane. The triangulation Δ is suitable for interpolation by $S_q^1(\Delta), q \geq 3$, respectively $S_q^2(\Delta), q \geq 5$.

Construction of the Triangulation. Let a set V of finitely many distinct points in $\mathbb{R}^2$ be given. We assume that V contains sufficiently many points. The triangulation Δ is constructed inductively as follows.

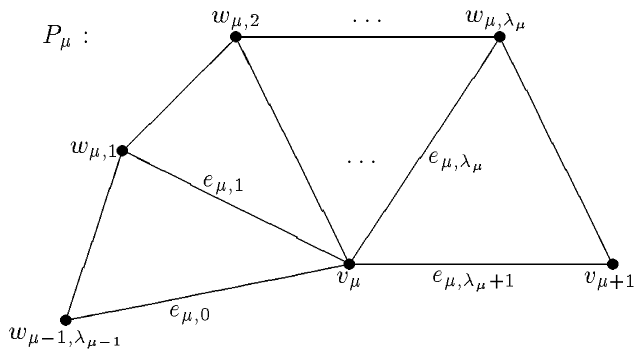

Fig. 4.1. Adding a polyhedron P_μ.

In the first step, we choose three points $v_1, v_2, v_3 \in V$ such that no point of V lies in the interior of the triangle formed by v_1, v_2, v_3. We assume that for a given subset $\tilde{V}$ of V, a simply connected triangulation $\tilde{\Delta}$ is already constructed with vertices in $\tilde{V}$. For simplicity, we denote the vertices on the boundary of $\tilde{\Delta}$ again by $v_1, \ldots, v_n$ (in clockwise order). For $\mu = 1, \ldots, n$, we choose points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu} \in V \setminus \tilde{V}$, $\lambda_\mu \geq 1$ (in clockwise order) such that no point of $V \setminus \tilde{V}$ lies in the interior of the polyhedron P_μ formed by the points $v_\mu, w_{\mu-1,\lambda_{\mu-1}}, w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}, v_{\mu+1}$, where $w_{0,\lambda_0} := v_n$ and $v_{n+1} := w_{1,1}$. We connect the points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$ with v_μ by line segments and denote the edges of P_μ with endpoint v_μ by $e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}$ (in clockwise order). We choose enough points $w_{\mu,1}, \ldots, w_{\mu,\lambda_\mu}$ such that $\lambda_\mu \geq 2$ if two edges in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$ have the same slope. Analogously, we choose $\lambda_\mu \geq 3$ if an edge in $\{e_{\mu,1}, \ldots, e_{\mu,\lambda_\mu}\}$ has the same slope as $e_{\mu,0}$ and a further edge in $\{e_{\mu,1}, \ldots, e_{\mu,\lambda_\mu}\}$ has the same slope as $e_{\mu,\lambda_\mu+1}$.

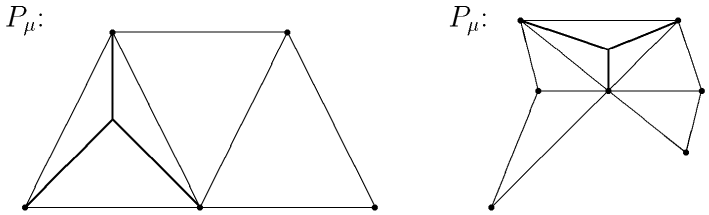

Fig. 4.2. Subdividing a triangle.

For the case when $r = 2$, exactly one triangle of P_μ has to be subdivided into three subtriangles if there do not exist four consecutive edges with different slopes in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$. This means that we use a Clough-Tocher split only in this case. Here, we subdivide a triangle of P_μ which has an edge $e_{\mu,\nu}$ with slope different from all other edges in $\{e_{\mu,0}, \ldots, e_{\mu,\lambda_\mu+1}\}$, or an arbitrary triangle of P_μ, if there does not exist such an edge (see Figure 4.2). We subdivide this triangle such that we obtain four consecutive edges with endpoint v_μ which have different slopes.

If there exist sufficiently many points such that for each $\mu \in \{1, \ldots, n\}$ a polyhedron P_μ with the above properties can be added, we obtain a larger triangulation. If for some $\mu \in \{1, \ldots, n\}$, such a polyhedron cannot be added, we choose a point from $V \setminus \tilde{V}$ and add a triangle with vertex v_μ which has exactly one common edge with the given subtriangulation and so forth. By proceeding with this method, we finally obtain a triangulation Δ with the points of V as vertices. Note that the polyhedrons can be chosen such that a natural triangulation is obtained.

Construction of Interpolation Sets. In the following, we construct Hermite interpolation sets for $S_q^r(\Delta)$, where $q \geq 3$, if $r = 1$, and $q \geq 5$, if $r = 2$. The construction of Hermite interpolation sets is inductive and simultaneous with the construction of the triangulation.

We only have to describe some basic Hermite interpolation conditions. For doing this, as in Section 2, we denote by $D_e f$ the directional derivative along the edge e. Let $T \in \Delta$ be an arbitrary triangle with vertices z_1, z_2, z_3 and denote by e_k the edge $[z_k, z_{k+1}]$, $k = 1, 2, 3$, where $z_4 = z_1$. For $r = 1$, we impose exactly one of the following conditions on the polynomial piece $p = s|_T \in \tilde{\Pi}_q$, where $s \in S_q^1(\Delta)$.

Condition Q: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q.$

Condition A_1: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 2.$

Condition B_1: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 3,$ and
$D_{e_2}^\alpha D_{e_3}^\beta p(z_3) = D_{e_2}^\alpha D_{e_3}^\beta f(z_3),\ 1 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta = q - 2.$

Condition D_1: $D_{e_1}^\alpha D_{e_2}^\beta p(z_2) = D_{e_1}^\alpha D_{e_2}^\beta f(z_2),\ \beta = 2, \ldots, q - 2 - \alpha,$
$\alpha = 0, \ldots, q - 4.$

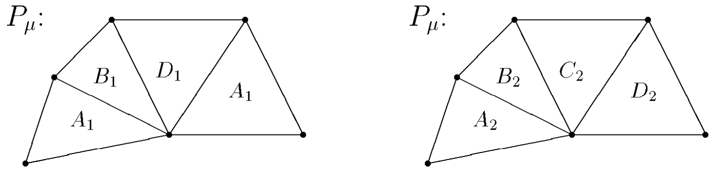

Fig. 4.3. Interpolation sets for $r = 1$, respectively $r = 2$.

We determine the polynomial piece of the interpolating spline $s \in S_q^1(\Delta)$ on the first triangle by condition Q. Moreover, for each polyhedron P_μ of our inductive construction, we determine the polynomial pieces on the corresponding triangles as follows. By the construction of Δ, three edges $e_{\mu,\nu}, e_{\mu,\nu+1}, e_{\mu,\nu+2}$ with different slopes exist. The polynomial pieces on the triangles of P_μ which do not have $e_{\mu,\nu+1}$ as an edge are determined by condition A_1 and the C^1-property of s. The remaining polynomial pieces are determined by condition B_1, respectively C_1 (See Figure 4.3). Moreover, if for some μ such a polyhedron P_μ cannot be added, we determine the polynomial piece on the triangle which is added by condition A_1.

The resulting set of Hermite interpolation conditions is denoted by $\mathcal{H}_1$. Note that we impose Hermite interpolation conditions only at the points of V. Similarly to Lagrange interpolation sets we speak of a `Hermite interpolation set` if for each sufficiently differentiable function f there exists a unique spline satisfying corresponding Hermite interpolation conditions.

Theorem 4.1. [26] *The set $\mathcal{H}_1$ is a Hermite interpolation set for $S_q^1(\Delta), q \geq 3$.*

For $r = 2$, we impose one of the following conditions on the polynomial piece $p = s|_T \in \tilde{\Pi}_q$, where $s \in S_q^2(\Delta)$.

Condition Q: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q.$

Condition A_2: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 3.$

Condition B_2: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 4$, and
$D_{e_2}^\alpha D_{e_3}^\beta p(z_3) = D_{e_2}^\alpha D_{e_3}^\beta f(z_3),\ 1 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta = q - 3.$

Condition C_2: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 4$, and
$D_{e_2}^\alpha D_{e_3}^\beta p(z_3) = D_{e_2}^\alpha D_{e_3}^\beta f(z_3),\ 2 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta = q - 3.$

Condition D_2: $D_{e_1}^\alpha D_{e_2}^\beta p(z_2) = D_{e_1}^\alpha D_{e_2}^\beta f(z_2),\ \beta = 3, \ldots, q - 3 - \alpha,$
$\alpha = 0, \ldots, q - 6.$

If a triangle is subdivided, we need the following additional condition.

Condition $\tilde{C}_2$: $D_x^\alpha D_y^\beta p(z_3) = D_x^\alpha D_y^\beta f(z_3),\ 0 \leq \alpha,\ 0 \leq \beta,\ \alpha + \beta \leq q - 4$, and
$D_{e_2}^\alpha D_{e_3}^\beta p(z_3) = D_{e_2}^\alpha D_{e_3}^\beta f(z_3),\ 2 \leq \alpha, 2 \leq \beta, \alpha + \beta = q - 3.$

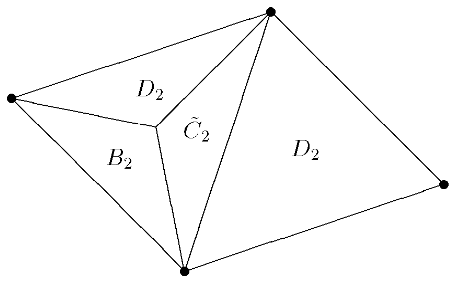

Fig. 4.4. Interpolation conditions for a subdivided triangle.

If no triangle of P_μ has to be subdivided, then by the construction of Δ, four edges $e_{\mu,\nu}, e_{\mu,\nu+1}, e_{\mu,\nu+2}, e_{\mu,\nu+3}$ with different slopes exist. In this case, the polynomial pieces of the interpolating spline $s \in S_q^2(\Delta)$ on the triangles of P_μ which do not have $e_{\mu,\nu+1}$, respectively $e_{\mu,\nu+2}$ as an edge are determined by condition A_2 and the C^2-property of s. The remaining polynomial pieces are determined by condition B_2, C_2 respectively D_2 (See Figure 4.3). If a triangle T with edges $e_{\mu,\nu}, e_{\mu,\nu+1}$ of P_μ is subdivided, then by construction of Δ the edges $e_{\mu,\nu}, e_{\mu,\nu+1}, e_{\mu,\nu+2}$ have different slopes. In this case, the polynomial pieces on the triangles of P_μ which do not have $e_{\mu,\nu+1}$ as an edge are determined by condition A_2. The four remaining polynomial pieces are determined by condition B_2, $\tilde{C}_2$ and D_2 (see Figure 4.4). Moreover, if for some μ such a polyhedron P_μ cannot be added, we determine the polynomial piece on the triangle which is added by condition A_2.

The resulting set of Hermite interpolation conditions is denoted by $\mathcal{H}_2$. Note that we only impose Hermite interpolation conditions at the points of V and the subdividing points.

Theorem 4.2. [26] *The set $\mathcal{H}_2$ is a Hermite interpolation set for $S_q^2(\Delta), q \geq 5$.*

Remark 4.3. Our method can also be used to construct Lagrange interpolation sets for $S_q^r(\Delta)$, where $q \geq 3$, if $r = 1$, and $q \geq 5$, if $r = 2$. For doing this, we choose distinct points lying on certain line segments in $T, T \in \Delta$. For details see [26].

Remark 4.4. By using Bézier-Bernstein techniques, we can show that the total number of interpolation conditions chosen by our method is equal to the dimension of $S_q^r(\Delta)$, where $q \geq 3$, if $r = 1$, and $q \geq 5$, if $r = 2$ (cf. [26]).

Remark 4.5. The interpolating spline is computed by passing from one triangle to the next and by solving several small systems instead of one large system. Therefore, the complexity of the algorithm is $\mathcal{O}(N)$, where N is the number of triangles.

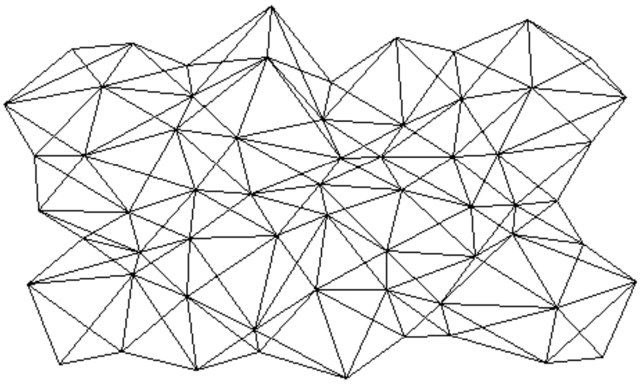

Fig. 4.5. A convex quadrangulation with diagonals.

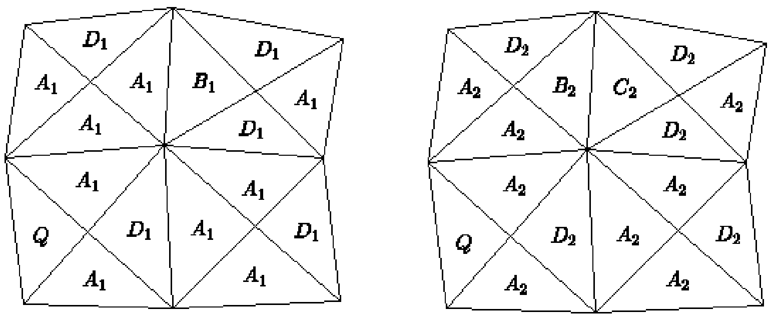

Fig. 4.6. Interpolation sets for splines on convex quadrangulations.

Remark 4.6. Our method can also be applied to certain classes of given triangulations, namely convex quadrangulations with diagonals. These are triangulations formed by closed convex quadrangles and their diagonals, where the intersection of any two quadrangles is empty, a common vertex or a common edge (see Figure 4.5). For such a triangulation, the distribution of interpolation conditions is indicated in Figure 4.6. In this case no triangle has to be subdivided (cf. [25]).

5 Hermite Interpolation with Optimal Approximation Order

We now describe a Hermite interpolation operator that assigns to every function $f \in C^{2r}(\Omega)$ a spline $s_f \in S_q^{r,\rho}(\Delta)$, where $S_q^{r,\rho}(\Delta)$ is the superspline subspace of

$S_q^r(\Delta)$, $q \ge 3r+2$,

$$S_q^{r,\rho}(\Delta) := \{s \in S_q^r(\Delta) : s \in C^\rho(v) \text{ for all vertices } v \text{ of } \Delta\},$$

with $\rho = r + \left[\frac{r+1}{2}\right]$. (The dimension of $S_q^{r,\rho}(\Delta)$ is given in [14].) Since restrictions of a spline $s \in S_q^{r,\rho}(\Delta)$ to every triangle of Δ are polynomials, we are allowed to use derivatives of order greater than ρ, but in this case a particular triangle $T \in \Delta$ has to be chosen so that the derivative information comes from $s|_T$.

Let $f \in C^{2r}(\Omega)$. We impose on a spline $s_f \in S_q^{r,\rho}(\Delta)$ the following Hermite interpolation conditions, that fall into three groups corresponding to all vertices, edges and triangles of Δ, respectively.

1) Given any vertex v of Δ, let $T_v^1, \ldots, T_v^n$ be all triangles attached to v and numbered counterclockwise (starting from a boundary triangle if v is a boundary vertex). Denote by e_i the common edge of T_v^{i-1} and T_v^i, $i = 2, \ldots, n$. If v is an interior vertex, $e_1 = e_{n+1}$ denote the common edge of T_v^1 and T_v^n. Otherwise, e_1 and e_{n+1} are the boundary edges (attached to v) of T_v^1 and T_v^n respectively. As in Section 2, we denote by $D_{e_i} f(v)$ the directional derivative of f along edge e_i. If $\alpha + \beta > \rho$, then we set $D_{e_i}^\alpha D_{e_{i+1}}^\beta s_f(v) := D_{e_i}^\alpha D_{e_{i+1}}^\beta (s_f|_{T_v^i})(v)$. For every vertex v in Δ the following conditions are imposed on $s_f \in S_q^{r,\rho}(\Delta)$:

- $D_x^\alpha D_y^\beta s_f(v) = D_x^\alpha D_y^\beta f(v)$ for all $(\alpha,\beta) \in A_1$, where

$$A_1 := \{(\alpha,\beta) \in \mathbb{Z}^2 : \ \alpha \ge 0, \ \beta \ge 0, \ \alpha+\beta \le \rho\},$$

- $D_{e_i}^\alpha D_{e_{i+1}}^\beta s_f(v) = D_{e_i}^\alpha D_{e_{i+1}}^\beta f(v)$ for all $(\alpha,\beta) \in A_2$, where

$$A_2 := \{(\alpha,\beta) \in \mathbb{Z}^2 : \ \alpha \le r, \ \beta \le r, \ \alpha+\beta \ge \rho+1\},$$

  and for each $i \in \{1, \ldots, n\}$ such that e_i is nondegenerate at v,

- $D_{e_i}^\alpha D_{e_{i+1}}^\beta s_f(v) = D_{e_i}^\alpha D_{e_{i+1}}^\beta f(v)$ for all $(\alpha,\beta) \in A_3$, where

$$A_3 := \{(\alpha,\beta) \in \mathbb{Z}^2 : \ \alpha \ge r+1, \ 2\alpha+\beta \le 3r+1, \ \alpha+\beta \ge \rho+1\},$$

  and for each $i \in \{1, \ldots, n\}$ such that e_i is degenerate at v,

- $D_{e_1}^\alpha D_{e_2}^\beta s_f(v) = D_{e_1}^\alpha D_{e_2}^\beta f(v)$ and $D_{e_{n+1}}^\alpha D_{e_n}^\beta s_f(v) = D_{e_{n+1}}^\alpha D_{e_n}^\beta f(v)$ for all $(\alpha,\beta) \in A_3$ if v is a boundary vertex, and
- $D_{e_1}^\alpha D_{e_2}^\beta s_f(v) = D_{e_1}^\alpha D_{e_2}^\beta f(v)$ for all $(\alpha,\beta) \in A_2$ if v is a singular vertex.

2) On every edge e of Δ, with vertices v' and v'', we choose points

$$z_e^{\mu,i} := v' + \frac{i}{\kappa_\mu + 1}(v'' - v'), \quad i = 1, \ldots, \kappa_\mu, \quad \mu = 0, \ldots, r,$$

where $\kappa_\mu := q - 3r - 1 - (r-\mu) \bmod 2 = q - 2r - 1 - \mu - 2\left[\frac{r+1-\mu}{2}\right]$, and impose on $s_f \in S_q^{r,\rho}(\Delta)$ the following conditions:

- $D^{\mu}_{e^\perp} s_f(z_e^{\mu,1}) = D^{\mu}_{e^\perp} f(z_e^{\mu,1}), \ldots, D^{\mu}_{e^\perp} s_f(z_e^{\mu,\kappa_\mu}) = D^{\mu}_{e^\perp} f(z_e^{\mu,\kappa_\mu})$ for all $\mu = 0, \ldots, r$, where $D_{e^\perp}$ denotes differentiation in the direction orthogonal to e.

3) On every triangle $T \in \Delta$, with vertices v', v'' and v''', we choose uniformly spaced points

$$z_T^{i,j,k} := (iv' + jv'' + kv''')/q\,, \quad i + j + k = q,$$

and impose on $s_f \in S_q^{r,\rho}(\Delta)$ the following conditions:

- $s_f(z_T^{i,j,k}) = f(z_T^{i,j,k})$ for all i, j, k such that $i+j+k = q$ and $r < i, j, k < q-2r$.

Theorem 5.1. [13] *Let $r \geq 1$, $q \geq 3r+2$ and $\rho = r + \left[\frac{r+1}{2}\right]$. Given $f \in C^{2r}(\Omega)$, there exists a unique spline $s_f \in S_q^{r,\rho}(\Delta)$ satisfying the above Hermite interpolation conditions. Moreover, if $f \in C^m(\Omega)$ ($m \in \{2r, \ldots, q+1\}$) and $T \in \Delta$, then*

$$||D_x^\alpha D_y^\beta (f - s_f)||_{L_\infty(T)} \leq K\, h_T^{m-\alpha-\beta} \max_{0 \leq m' \leq m} ||D_x^{m'} D_y^{m-m'} f||_{C(T)},$$

for all $\alpha, \beta \geq 0$, $\alpha + \beta \leq m$, where h_T is the diameter of T and K is a constant which depends only on r, q and the smallest angle θ_Δ in Δ.

The following new characterization of C^r smoothness across a common edge of two polynomial patches plays an essential role in the proof of Theorem 5.1.

Theorem 5.2. [13] *Let T_1 and T_2 be two triangles sharing a common edge $e = [v_1, v_2]$, and let e_i be the edge of T_i attached to v_1 and different from e, $i = 1, 2$. Suppose a piecewise polynomial function s is defined on $T_1 \cup T_2$ as follows*

$$s|_{T_i} = p_i \in \Pi_q\,, \quad i = 1, 2\,.$$

Then $s \in C^r(T_1 \cup T_2)$, for some $r \leq q$, if and only if

$$\tau_1^\alpha D_{e_2}^\alpha D_e^{\gamma-\alpha} p_2(v_1) = \sum_{\beta=0}^{\alpha} (-1)^\beta \binom{\alpha}{\beta} \sin^{\alpha-\beta}(\theta_1 + \theta_2) \tau_2^\beta D_{e_1}^\beta D_e^{\gamma-\beta} p_1(v_1),$$

for all $\alpha = 0, \ldots, r$ and $\gamma = \alpha, \ldots, q$, where

$$\tau_i = \begin{cases} \sin\theta_i, & \text{if } e_1 \text{ and } e_2 \text{ are noncollinear,} \\ 1, & \text{otherwise,} \end{cases}$$

and θ_i is the angle between e and e_i, $i = 1, 2$.

It follows from Theorem 5.1 that the fundamental functions $s_1, \ldots, s_N$ of the above Hermite interpolation scheme form a basis for $S_q^{r,\rho}(\Delta)$. We note that a basis for this space has been constructed in [14] by using Bernstein-Bézier techniques. Although there exists some interrelation between two bases, particularly, the supports of basis functions are the same, the minimal determining set of [14] cannot be transformed by standard Bernstein-Bézier arguments into a Hermite interpolation scheme of our type.

The next theorem lists some of useful properties of our basis.

Theorem 5.3. [13] *The fundamental functions $s_1, \ldots, s_N$ form a basis for $S_q^{r,\rho}(\Delta)$ such that*

1) $\{s_1, \ldots, s_N\}$ *is locally linearly independent, i.e., for every open $B \subset \Omega$ the subsystem $\{s_i : B \cap \operatorname{supp} s_i \neq \emptyset\}$ is linearly independent on B,*

2) $\{s_1, \ldots, s_N\}$ *is least supported, i.e., for every basis $\{b_1, \ldots, b_N\}$ of $S_q^{r,\rho}(\Delta)$ there exists a permutation π of $\{1, \ldots, N\}$ such that*

$$\operatorname{supp} s_i \subset \operatorname{supp} b_{\pi(i)}, \quad \textit{for all} \quad i = 1, \ldots, N,$$

3) $\operatorname{supp} s_i$, $i = 1, \ldots, N$, *is either a triangle or the union of some triangles sharing one common vertex, and*

4) *the corresponding normalized basis $\{s_1^*, \ldots, s_N^*\}$, with*

$$s_i^* := \|s_i\|_{L_\infty(\Omega)}^{-1} s_i, \quad i = 1, \ldots, N,$$

is stable in the sense that

$$K_1 \max_i |a_i| \leq \|\sum_{i=1}^{N} a_i s_i^*\|_{C(\Omega)} \leq K_2 \max_i |a_i|,$$

where K_1 and K_2 depend only on r, q, θ_Δ and some measure of "near-degeneracy" of nondegenerate edges in Δ.

Remark 5.4. Theorem 5.1 provides a new proof of the optimal approximation order of $S_q^r(\Delta)$, $q \geq 3r+2$. Previous results on this subject were given in [3, 7, 8, 16]. As in [7, 16], the constant K that appears in Theorem 5.1 depends only on r, q and the smallest angle in Δ, and, therefore, does not grow for triangulations that contain near-singular vertices. Moreover, in contrast to quasi-interpolation methods of [7, 16], we show that optimal approximation order can be achieved by using Hermite interpolation.

Remark 5.5. According to Theorem 5.3, 2), our basis is best possible for the space $S_q^{r,\rho}(\Delta)$ in regard to the size of the supports of the basis functions. It shares this property with the basis constructed in [14]. The bases in [7, 16] fail to be least supported, but they have the advantage that stability constants K_1, K_2 depend only on the smallest angle in the triangulation while in our construction they also depend on the sums of pairs of adjacent angles.

6 Numerical Results

Finally, we give some numerical results for the interpolation methods of Section 3 and Section 4. We interpolate Franke's test function

$$\begin{aligned} f(x,y) &= \tfrac{3}{4} e^{-\frac{(9x-2)^2+(9y-2)^2}{4}} + \tfrac{3}{4} e^{-\frac{(9x+1)^2}{49} - \frac{(9y+1)}{10}} + \tfrac{1}{2} e^{-\frac{(9x-7)^2+(9y-3)^2}{4}} \\ &\quad - \tfrac{1}{5} e^{-(9x-4)^2-(9y-7)^2} \end{aligned}$$

Δ:

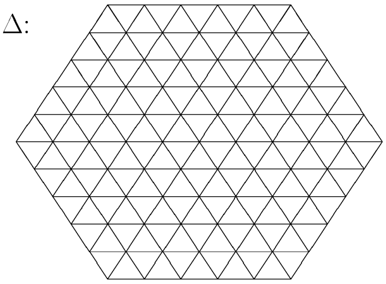

Fig. 6.1. Triangulation Δ.

by splines on a domain Ω with $[0,1]\times[0,1]\subseteq\Omega$. First, let Δ be a triangulation as in Figure 6.1. Obviously, Δ is of nested-polygon type.
Our results for the Hermite interpolating spline $s_f \in S_3^1(\Delta)$ are as follows :

$$[S_3^1 \mid 93 \mid 1.38 * 10^{-1}], \quad [S_3^1 \mid 291 \mid 1.64 * 10^{-2}],$$
$$[S_3^1 \mid 1011 \mid 1.20 * 10^{-3}], \quad [S_3^1 \mid 3747 \mid 1.61 * 10^{-4}],$$
$$[S_3^1 \mid 14403 \mid 2.03 * 10^{-5}],$$

where we set

$$[S_q^r \mid \text{number of interpolation conditions} \mid \text{error } \|f - s_f\|_\infty].$$

Now, let Δ be a triangulation that results from a given Δ^2-partition deformed by a randomizer (see Figure 4.5). Our results for the Hermite interpolating spline $s_f \in S_q^r(\Delta)$ are as follows :

$$[S_3^1 \mid 168 \mid 4.51 * 10^{-2}], \quad [S_3^1 \mid 583 \mid 1.05 * 10^{-2}],$$
$$[S_3^1 \mid 2163 \mid 8.00 * 10^{-4}], \quad [S_3^1 \mid 8323 \mid 9.06 * 10^{-5}],$$
$$[S_4^1 \mid 388 \mid 7.09 * 10^{-2}], \quad [S_4^1 \mid 1423 \mid 4.01 * 10^{-3}],$$
$$[S_4^1 \mid 5443 \mid 1.36 * 10^{-4}], \quad [S_4^1 \mid 21283 \mid 5.70 * 10^{-6}],$$
$$[S_5^1 \mid 708 \mid 4.23 * 10^{-2}], \quad [S_5^1 \mid 2663 \mid 2.12 * 10^{-3}],$$
$$[S_5^1 \mid 10323 \mid 2.61 * 10^{-5}], \quad [S_5^1 \mid 40643 \mid 7.17 * 10^{-7}],$$
$$[S_7^2 \mid 993 \mid 8.23 * 10^{-3}], \quad [S_7^2 \mid 3678 \mid 3.26 * 10^{-4}],$$
$$[S_7^2 \mid 14148 \mid 1.71 * 10^{-6}], \quad [S_7^2 \mid 55488 \mid 3.53 * 10^{-8}],$$
$$[S_8^2 \mid 1473 \mid 1.79 * 10^{-3}], \quad [S_8^2 \mid 5538 \mid 1.36 * 10^{-5}],$$
$$[S_8^2 \mid 21468 \mid 2.98 * 10^{-7}], \quad [S_8^2 \mid 84258 \mid 7.54 * 10^{-9}].$$

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